DERIVED LANGLANDS VI: MONOMIAL RESOLUTIONS AND 2-VARIABLE L-FUNCTIONS

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Contents

1. $L(s, \mathrm{Ind}_H^{\hat{G}_F}(\phi), \pi)$ for local fields

Due to time pressure (see footnote 1) this essay, which is part of a series consisting of a book [64] and ([65], [66], [67], [68]), has a number of speculative constructions, which I do not have time to provide the rigourous details. The main constructions appear in §1 and §2, the other sections being largely for reference.

Let G be a (usually connected) reductive algebraic group defined over a global field F. Therefore F is an algebraic number field or a function field in one variable over a finite field.

Often I shall be concerned with the points of G over some local field given the completion of F at a non-Archimedean prime of F. In this case, now writing F for its local completion, suppose that K is a (usually finite) Galois extension of F and that G is a quasi-split group over F which splits over K.

I am going to use (doing my best to given page and line references for the terminology) the notation and conventions of [43]. That is a rather tall order for the reader, should one exist, but the notation in [43] is very technical

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and elaborate and is explained at more length than I can manage in the time available[1]

Let K/F be a Galois extension of non-Archimedean local fields with Galois group $\mathrm{Gal}(K/F)$.

The main construction of this essay will be in the context of the following result.

Theorem 1.1. *(*[43] *)*

There is a Chevalley lattice in the Lie algebra of G whose stabiliser U_K is invariant under $\mathrm{Gal}(K/F)$. U_K is self-normalising. Moreover, $G_K = B_K U_K$, $H^1(\mathrm{Gal}(K/F); U_K) = \{1\}$ and $H^1(\mathrm{Gal}(K/F); B_K \bigcap U_K) = \{1\}$. If we choose two such Chevalley lattices with stabilisers U_K and U'_K then U_K and U'_K are conjugate in G_K.

The notation for Theorem 1.1 is given on ([43] pp.29 final paragraph). Suppose that K/F is a (not necessarily unramified for this essay) extension of local fields and G is a quasi-split group over F which splits over K. Let B be a Borel subgroup of G and T a Cartan subgroup of B both of which are defined over F. Let v be the valuation on K. It is a homomorphism from K^* whose kernel is the group of units. $\mathcal{O}_K^*$. If $t \in T_F$ let $v(t) \in \hat{L}$ be defined by $< \lambda, v(t) >= v(\lambda(t))$ for all $\lambda \in L$, the group of rational characters of T ([43] p,22, line -6).

The dual group $\hat{G}$ of G and the Galois action on $\mathrm{Gal}(K/F)$ on it are defined in ([43] p.22 §2 to p.26 line 10) and, as hinted at above, the notation is quite involved but the constructions are straightforward enough. This material enables one to define ([43] p.26 line 10) $\hat{G}_F$[2], which is the semi-direct product of the Galois group with $\hat{G}$.

Suppose that ρ is a complex analytic representation of the semi-direct product $\hat{G}_F$ ([43] p.34 line 1)[3] and that π is an irreducible unitary representation of G_F on $\mathcal{H}$ whose restriction to U_F contains the trivial representation (i.e. $\mathcal{H}^{U_F}$ is one-dimensional.

If $C_c(G_F, U_F)$ is the Hecke algebra ([43] p. 30 line -7) of compactly supported functions f such that $f(gu) = f(g) = f(ug)$ for all $u \in U_F, g \in G_F$. There is a representation of $C_c(G_F, U_F)$ on the subspace $\mathcal{H}^{U_F}$ which gives a homomorphism χ from $C_c(G_F, U_F)$ into the ring of complex numbers ([43] p.33 line 6). The Galois properties of this χ determine a well-defined conjugacy class in the semi-direct product $\hat{G}_F$ - denoted by $t\sigma_F$ in ([43] p.34).

[1] A deteriorating health problem over which I am not in control [50].

[2] Correction: On rereading [43] more carefully I now understand that when admissible representations are complex $\hat{G}_F$ is a complex analytic Lie group with a combinatorial action by the Galois group. When the represenations are over k, an algebraically closed local field of characteristic zero, $\hat{G}_F$ is a k-analytic Lie group with combinatorial Galois action.

[3] In [43] K/F is taken to be an unramified extension of local fields, which is sufficient but immaterial.

Therefore $\rho(t\sigma_F)$ makes sense, up to conjugate, and Langlands defines the 2-variable L-function in tis case by the formula ([43] p.34 line 4)

$$L(s,\rho,\pi) = \frac{1}{\det(1-\rho(t\sigma_F)|\pi_F|^s)}$$

where π_F generates the maximal ideal of $\mathcal{O}_F$.

Note that the "det" in Langlands definition above is legitimatised in the sense of ([3] §Theorem 3.1). A semi-simple element a in the socle of a Banach algebra has an associated determinant, $\det(1-a)$ given by the formula of ([3] §Theorem 3.1). The Banach algebra involved can be taken to be any completion containing $\rho(t\sigma_F)$.

The same legitimisation, this time taking place in the $n \times n$ matrix ring with entries of the type $\rho(t_{i,j}\sigma_F)$ (see next paragraph) will be required in the explanation of the definition of

$$L(s, \mathrm{Ind}_H^{\hat{G}_F}(\phi), \pi)$$

where H is a subgroup of the semi-direct product $\hat{G}_F$ containing the semi-direct product of $Z(\hat{G}_K)$ with $\mathrm{Gal}(K/F)$, modulo which it is compact open, and ϕ is a continuous complex-values character on H.

Firstly I believe that the example of §4 is typical and that $\mathrm{Ind}_H^{\hat{G}_F}(\phi)^U$ is finite-dimensional. Choosing a basis $v_1, \ldots, v_n$ gives, by Langlands construction an $n \times n$ "matrix" M of examples $t_{i,j}\sigma_F$ in the semi-direct product $\hat{G}_F$ on which the effect of making different choices is to conjugate the matrix - elementwise - by an element of $\hat{G}_F$. Let ρ^{I_F} denote the representation of $\hat{G}_F$ given by the subspace of ρ fixed by the semi-direct product of the Galois group with the decomposition group of i_F of π_F.

Modulo the legitimisation of "det" my definition if the L-function is given by

$$L(s, \mathrm{Ind}_H^{\hat{G}_F}(\phi), \pi) = \frac{1}{\det(1-\rho^{I_F}(M)|\pi_F|^s)}.$$

I expect this definition to be well-defined, to be bi-multiplicative in the variables ρ and π and whose adelic Euler product should enjoy the sort of inductivity properties which are satisfied by the Artin L-function which are recapitulated in §§6-10.

2. Expectations of $L(s,\rho,\pi)$ via monomial resolutions

For ρ and π admissible representations in the local field situation of §1 suppose that

$$\ldots \longrightarrow M_i \longrightarrow M_{i-1} \ldots \longrightarrow M_0 \longrightarrow V \longrightarrow 0$$

is a monomial resolution ([64], [65] §§8-10) then we are entitled, from §1, to a 2-variable L-function $L(s, M_i, \pi)$ defined as the product of the 2-variable L-functions $L(s, \mathrm{Ind}_{H_{i,j}}^{\hat{G}_F}(\phi), \pi)$ such that $M_i = \oplus_j \underline{\mathrm{Ind}}_{H_{i,j}}^{\hat{G}_F}(\phi)$ in the monomial category.

When the representations are complex the monomial resolution is of "finite type" - a consequence of monomial resolutions of finite dimmensional representations of finite groups being actually finite [64]. As a consequence I expect this definition of $L(s, \rho, \pi)$ to coincide with the definition of ([43] pp.29-34) and to enjoy all the analytic properties of that example.

In ([65] §4) the notion of $\mathcal{M}_{cmc,\underline{\phi}}(G)$-admissibility is introduced. It is particularly interesting in the di-p-adic situation (where the local field is p-adic and the representations are defined over the algebraic closure of $\mathbb{Q}_p$.

In this context the monomial resolution is not necessarily of finite type, as far as I know at the moment, nevertheless I am still optimistic about the following conjecture.

Conjecture 2.1.

The multiplicative Euler characteristic of 2-variable L-functions $L(s, M_i, \pi)$ defines a well-defined and analytically well-behaved L-function $L(s, \rho, \pi)$.

When our Galois extension K/F is an extension of global fields, it is explained in [64] how to use the adelic Tensor Product Theorem ([64] Theorem 1.21) to define adelic monomial resolutions of automorphic representations. In this situation there is a simple reduction to the case in which the finite Galois group is soluble, which is based on the fact that the subgroups involved in the monomial resolution of any finite-dimensional of a finite group over any algebraically closed field of characteristic zero may be taken to be M-groups ([62] Proposition 2.1.17 p.30).

Recall that a finite group G is nilpotent if and only if it has a lower central series

$$\{1\} = Z_0 \lhd Z_1 = Z(G) \lhd Z_2 \lhd \ldots \lhd Z_n = G$$

exists such that $Z_{i+1}/Z_i = Z(G/Z_i)$ for all i. In particular nilpotent groups are M-groups - each irreducible representation is induced from a 1-dimensional character of a subgroup. Since nilpotent groups are the product of their Sylow p-subgroups a subgroup of a nilpotent group is again nilpotent - but not so for M-groups.

An M-group is soluble [37]. The derived series of G is

$$G^0 = G, G^1 = [G^0, G^0] = [G, G], \text{the commutator subgroup of G}$$

and $G^n = [G^{n-1}, G^{n-1}]$ for $n = 1, 2, 3, \ldots$. A soluble group G is one for which $G^n = \{1\}$ for some n. Subgroups of soluble groups are soluble since $J < G$ implies $[J, J] < [G, G]$. We begin by writing the trivial one-dimensional representation of the Galois group as a sum of iduced monomial representations $\mathrm{Ind}_H^G(\phi)$ with H an M-subgroup of G. We can inflate this relation to the semi-direct prodct and multiply the monomial resolution, as a complex of representation, with this formula for 1.

Recall also that the product in $R_+(G)$ is given by a double coset formula ([62] p.68, Exercise 2.5.7)

$$(K,\phi)^G \cdot (H,\psi)^G$$

$$= \sum_{w \in K\backslash G/H} (w^{-1}Kw \bigcap H, w^*(\phi)\psi)^G.$$

Suppose, abbreviating Gal(K/F) to Gal, we have a Galois semi-direct product Gal $\propto G$ with $(K,\phi)^{\mathrm{Gal}\propto G} \in R_+(\mathrm{Gal} \propto G)$ and if $(H,\psi)^{\mathrm{Gal}}$ and $\lambda : \mathrm{Gal} \propto G \longrightarrow \mathrm{Gal}$ is the projection then $(\lambda^{-1}(H), \psi\lambda)^{\mathrm{Gal}\propto G}$. Therefore

$$(K,\phi)^{\mathrm{Gal}\propto G} \cdot (\lambda^{-1}(H), \psi\lambda)^{\mathrm{Gal}\propto G}$$

$$= \sum_{w \in K\backslash \mathrm{Gal}\propto G/H\propto G} (w^{-1}Kw \bigcap H, w^*(\phi)\psi)^G.$$

If $w = (z,g) \in \mathrm{Gal} \propto G$ and $(y,g_1) \in H \propto G$ then $(z,g)(y,g_1) = (zy, gz(g_1))$so we may take $w = (z,1) \in K \bigcap \mathrm{Gal}\backslash\mathrm{Gal}/H$ and $w^{-1}Kw \bigcap H \propto G \subset H \propto G$.

This procedure reduced the Galois semi-direct products to ones involving solution Galois groups.

3. Example of ([43] pp.29-34) when $G = GL_2$ and $K/F = \mathbb{F}_q/\mathbb{F}_p$

In this section I am going to study a finite group analogue of the combinatorial construction which will give us the 2-valued L-function by means of the monomial resolution of an admissible complex representation.

Consider the rings, under convolution, of complex valued functions

$$C(GL_2\mathbb{F}_p, \{u = \left(\begin{array}{cc} 1 & 0 \\ & \\ x & 1 \end{array}\right) \mid x \in \mathbb{F}_p\}) = \{f \mid f(ug) = f(g) = f(gu)\}$$

and

$$C(GL_2\mathbb{F}_q, \{u = \left(\begin{array}{cc} 1 & 0 \\ & \\ x & 1 \end{array}\right) \mid x \in \mathbb{F}_q\}) = \{f \mid f(ug) = f(g) = f(gu)\}$$

in the manner of ([43] pp.29-34).

Inside $GL_2\mathbb{F}_q$ take the subgroup

$$U_{\mathbb{F}_q} = \{ \left(\begin{array}{cc} \alpha & 0 \\ & \\ x & \alpha \end{array}\right) \mid x \in \mathbb{F}_q, \alpha \in \mathbb{F}_q^*\}.$$

I have chosen this group because it has the four properties of Theorem 1.1. Also

$$C(GL_2\mathbb{F}_q, U_{\mathbb{F}_q}) = C(GL_2\mathbb{F}_q, \{u = \left(\begin{array}{cc} 1 & 0 \\ & \\ x & 1 \end{array}\right) \mid x \in \mathbb{F}_q\}).$$

For the four properties, clearly $GL_2\mathbb{F}_q = B_{\mathbb{F}_q}U_{\mathbb{F}_q}$, $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ acts on $U_{\mathbb{F}_q}$ and $H^1(\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p); U_{\mathbb{F}_q}) = \{1\}$ by Hilbert 90 for $\mathbb{F}_q^*$ and $\mathbb{F}_q$ and similarly $H^1(\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p); B_{\mathbb{F}_q} \bigcap U_{\mathbb{F}_q}) = \{1\}$ since $B_{\mathbb{F}_q} \bigcap U_{\mathbb{F}_q} = \mathbb{F}_q^*$.

Now for the normaliser of $U_{\mathbb{F}_q}$. If

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & 0 \\ x & a \end{pmatrix} = \begin{pmatrix} b & 0 \\ y & b \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

we have

$$\alpha a + x\beta = b\alpha,\ \beta a = b\beta,\ \gamma a + \delta x = y\alpha + b\gamma,\ \delta a = y\beta + b\delta.$$

Therefore $\beta = 0$ or $a = b$. If $\beta = 0$ we have

$$\alpha a = b\alpha,\ \gamma a + \delta x = y\alpha + b\gamma,\ a = b$$

and $x = \frac{y\alpha}{\delta}$. If $a = b$ we have

$$x\beta = 0,\ \delta x = y\alpha,\ 0 = y\beta$$

so if $y \neq 0$ then $\beta = 0$ and we are in the same case as before. In general we have

$$\begin{pmatrix} a\alpha & 0 \\ \gamma a + \delta x & \delta a \end{pmatrix}$$

$$= \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & 0 \\ x & a \end{pmatrix}$$

$$= \begin{pmatrix} b & 0 \\ y & b \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}$$

$$= \begin{pmatrix} b\alpha & 0 \\ y\alpha + b\gamma & b\delta \end{pmatrix}$$

so $\alpha = \delta$ and $U_{\mathbb{F}_q}$ is equal to its own normaliser in $GL_2\mathbb{F}_q$.

A finite group example of the convolution algebra.

Let (π, V) be a finite-dimensional representation of a finite group G[4]. Write $\mathcal{H}$ for the space of functions from G to $\mathbb{C}$. If $\phi_1, \phi_2 \in \mathcal{H}$ define $\phi_1 * \phi_2 \in \mathcal{H}$ by

$$(\phi_1 * \phi_2)(g) = \sum_{h \in G} \phi_1(gh^{-1})\phi_2(h).$$

[4] Usually I shall consider the set of representation of G with a fixed central character $\underline{\phi}$ and the monomial resolution associated with this central character condition [64].

For $\phi \in \mathcal{H}$ define $\tilde{\pi}(\phi) \in \mathrm{End}_{\mathbb{C}}(V)$ by

$$\tilde{\pi}(\phi)(v) = \sum_{g \in G} \phi(g)\pi(g)(v).$$

Hence

$$\tilde{\pi}(\phi_1(\tilde{\pi}(\phi_2)(v))$$

$$= \tilde{\pi}(\phi_1)(\textstyle\sum_{g\in G} \phi_2(g)\pi(g)(v))$$

$$= \textstyle\sum_{g\in G} \phi_2(g)\pi(\phi_1(\pi(g)(v))$$

$$= \textstyle\sum_{g\in G} \phi_2(g) \sum_{\tilde{g}\in G} \phi_1(\tilde{g})(\pi(\tilde{g}(\pi(g)(v))$$

$$= \textstyle\sum_{g,\tilde{g}\in G} \phi_2(g)\phi_1(\tilde{g})(\pi(\tilde{g}g)(v)).$$

Now

$$\pi(\phi_1 * \phi_2)(v)$$

$$= \textstyle\sum_{g_1\in G} (\phi_1 * \phi_2)(g_1)\pi(g_1)(v)$$

$$= \textstyle\sum_{g_1,h\in G} \phi_1(h_1h^{-1})\phi_2(h)\pi(g_1)(v).$$

Setting $g = h$, $\tilde{g}g = g_1$ shows that

$$\pi(\phi_1 * \phi_2) = \pi(\phi_1) \cdot \pi(\phi_2).$$

Therefore $\tilde{\pi} : \mathcal{H} \longrightarrow \mathrm{End}(V)$ is a ring homomorphism.

Also $\mathcal{H} \cong \mathbb{C}[G]$ because if $f_g(x) = 0$ if $g \neq x$ and $f_g(g) = 1$ then

$$f_g * f_{g'} = f_{gg'}.$$

Now take $G = GL_2\mathbb{F}_p$ so that $C(GL_2\mathbb{F}_p, U_{\mathbb{F}_p}) \subset \mathcal{H}$ and if $\phi \in C(GL_2\mathbb{F}_p, U_{\mathbb{F}_p})$ then for

$$u = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$$

and $v = uv$ then

$$\pi(u)(\tilde{\pi}(\phi)(v)) = \sum_{g\in G} \phi(g)\pi(u)(\pi(g)(v)) = \sum_{g\in G} \phi(ug)\pi(ug)(v)) = \tilde{\pi}(\phi)(v)$$

so that $\tilde{\phi} : V^{\{\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \mid x\in\mathbb{F}_p\}} \longrightarrow V^{\{\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \mid x\in\mathbb{F}_p\}} = W$. Therefore we have a ring homomorphism from $C(GL_2\mathbb{F}_p, U_{\mathbb{F}_p})$ to the ring of endomorphisms of V which map W to itself.

The key step in ([43] pp. 29-34) is now to be in a situation where W is one-dimensional so that sending $\pi \in C(GL_2\mathbb{F}_p, U_{\mathbb{F}_p})$ to the scalar by which

$\tilde{\pi}(\phi)$ acts on W gives a ring homomorphism $C(GL_2\mathbb{F}_p, U_{\mathbb{F}_p}) \longrightarrow \mathbb{C}$. We shall not assume any such restriction.

Now let us consider some irreducibles of $GL_2\mathbb{F}_p$.

Suppose that V is the vector space on which we have the representation $V = R(\chi_1, \chi_2) = \mathrm{Ind}_{B_{\mathbb{F}_p}}^{GL_2\mathbb{F}_p}(\chi_1 \otimes \chi_2)$ with $\chi_2 \neq \chi_2 : \mathbb{F}_p^* \longrightarrow \mathbb{C}^*$ ([62] p.89).

Therefore

$$V^{U_{\mathbb{F}_p}} = \langle \sum_{x \in \mathbb{F}_p} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \otimes_{B_{\mathbb{F}_p}} 1 \rangle.$$

A basis for $C(GL_2\mathbb{F}_p, U_{\mathbb{F}_p})$ is

$$\phi_{\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}}$$

satisfying

$$\phi_{\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}}(g) = \begin{cases} 1 & \text{if } g \in B_{\mathbb{F}_p}, g = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \\ 0 & \text{if } g \in B_{\mathbb{F}_p}, g \neq \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}. \end{cases}$$

Therefore

$$\tilde{\pi}(\phi_{\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}})(\sum_{x\in\mathbb{F}_p} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \otimes_{B_{\mathbb{F}_p}} 1)$$

$$= \sum_{g\in GL_2\mathbb{F}_p} \phi_{\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}}(g)\pi(g)(\sum_{x\in\mathbb{F}_p} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \otimes_{B_{\mathbb{F}_p}} 1)$$

$$= p\sum_{g\in B_{\mathbb{F}_p}} \phi_{\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}}(g) \sum_{x\in\mathbb{F}_p} g \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \otimes_{B_{\mathbb{F}_p}} 1$$

$$= p\sum_{x\in\mathbb{F}_p} \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \otimes_{B_{\mathbb{F}_p}} 1$$

but

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{dx}{a+bx} & 1 \end{pmatrix} \begin{pmatrix} a+bx & b \\ 0 & \frac{da}{a+bx} \end{pmatrix}$$

so that

$$\tilde{\pi}(\phi_{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}})(\sum_{x\in\mathbb{F}_p} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \otimes_{B_{\mathbb{F}_p}} 1)$$

$$= p\sum_{x\in\mathbb{F}_p} \begin{pmatrix} 1 & 0 \\ \frac{dx}{a+bx} & 1 \end{pmatrix} \otimes_{B_{\mathbb{F}_p}} \chi_1(a+bx)\chi_2(\tfrac{da}{a+bx}).$$

Next consider a character $\Theta : \mathbb{F}_{p^2}^* \longrightarrow \mathbb{C}^*$ such that $\sigma_{\mathbb{F}_p}(\Theta) \neq \Theta$ and the associated Weil representation ([62] Chapter 3.1)

$$r(\Theta) : GL_2\mathbb{F}_p \longrightarrow GL_{p-1}\mathbb{C}.$$

It is irreducible and we have a short exact sequence of representations ([64] Example 3.4 pp. 316-317)

$$0 \longrightarrow \mathrm{Ind}_{\mathbb{F}_{p^2}^*}^{GL_2\mathbb{F}_p}(\Theta) \longrightarrow \mathrm{Ind}_H^{GL_2\mathbb{F}_p}(\Theta\otimes\Psi) \longrightarrow r(\Theta) \longrightarrow 0$$

where $H = \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in GL_2\mathbb{F}_p\}$ and $\Theta \otimes \Psi(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}) = \Theta(a)\Psi(b/a)$ where $\Psi : \mathbb{F}_{p^2} \stackrel{\text{Trace}}{\longrightarrow} \mathbb{F}_p \longrightarrow \mathbb{C}^*$, the final map being $x(\text{modulo } p) \mapsto (e^{\frac{2\pi\sqrt{-1}}{p}})^x$.

A basis for the vector subspace

$$(\mathrm{Ind}_{\mathbb{F}_{p^2}^*}^{GL_2\mathbb{F}_p}(\Theta))^{U_{\mathbb{F}_p}}$$

is given by, abbreviating $U_{\mathbb{F}_p}$ to U

$$v_g = \frac{1}{|U|}\Sigma_{u' \in U}\ u'g \otimes_{\mathbb{F}_{p^2}^*} 1 \text{ for } g \in U\backslash GL_2\mathbb{F}_p/\mathbb{F}_{p^2}^*$$

Note that $v_{gw} = (\Theta)(w)^{-1}$ for $w \in \mathbb{F}_{p^2}^*$.

A basis for the vector subspace

$$(\mathrm{Ind}_H^{GL_2\mathbb{F}_p}(\Theta \otimes \Psi))^{U_{\mathbb{F}_p}}$$

is given by, again abbreviating $U_{\mathbb{F}_p}$ to U

$$v_g = \frac{1}{|U|}\Sigma_{u \in U}\ ug \otimes_H 1 \text{ for } g \in U\backslash GL_2\mathbb{F}_p/H.$$

Note that $v_{gh} = (\Theta \otimes \Psi)(h)^{-1}$ for $h \in H$.

The dimensions of these spaces of U-invariants are $p = |U|$ and $p-1$ respectively while , of course, the dimension of the U-invariants of $r(\Theta)$ is one.

The construction therefore gives a map from $GL_2\mathbb{F}_p$ to $p \times p$ complex matrices and $(p-1)\times(p-1)$-complex metrices. The map to $\mathbb{C}$ given by $r(\Theta)$ is the quotient of the determinants of the H-matrix divided by the $\mathbb{F}_{p^2}^*$ matrix.

The final move in the ([43] pp.29-34) gives, from our data, a matrix of elements in the semi-direct product of the Galois group with the dual group of G. We are not going to do this step in this section, but if we were this entire matrix will be independent of choices up to conjugation by an element of the semi-direct product, as happens when the matrix is merely 1×1 in the construction of ([43] pp.29-34).

4. Haar integration on K

Example 4.1. *Some Haar measures on K*

Let r be a positive integer. A measure on $\mathcal{P}_K^{-r}$ is a family of functions $\phi_{K,r,n} : \mathcal{P}_K^{-r}/\mathcal{P}_K^n \longrightarrow \mathbb{C}$ for $n \geq N_0$ which satisfy

$$\phi_{K,r,n}(x + \mathcal{P}_K^n) = \sum_{y+\mathcal{P}_K^{n+1}\ |\ y \in x+\mathcal{P}_K^n} \phi_{K,r,n+1}(y + \mathcal{P}_K^{n+1}).$$

In this case the sum

$$I_{K,r,n}(f) = \sum_{x \in \mathcal{P}_K^{-r}/\mathcal{P}_K^n} f(x)\phi_{K,r,n}(x + \mathcal{P}_K^n)$$

is well-defined for each $n >> 0$ and independent of n. This is because f is locally constant so that there is an n such that $f(x)$ depends only on the coset $x + \mathcal{P}_K^n$ and in this case

$$I_{K,r,n+1}(f)$$

$$= \sum_{y \in \mathcal{P}_K^{-r}/\mathcal{P}_K^{n+1}} \ f(y)\phi_{K,r,n+1}(y + \mathcal{P}_K^{n+1})$$

$$= \sum_{x \in \mathcal{P}_K^{-r}/\mathcal{P}_K^{n}} \ f(x) \ \sum_{y + \mathcal{P}_K^{n+1} \ | \ y \in x + \mathcal{P}_K^n} \ \phi_{K,r,n+1}(y + \mathcal{P}_K^{n+1})$$

$$= I_{K,r,n}(f).$$

If, for example, we set $\phi_{K,r,n}(x + \mathcal{P}_K^n) = |\mathcal{O}_K/\mathcal{P}_K|^{(1/2)-n}$ for all $n \geq 0$ then

$$\begin{aligned} \sum_{y + \mathcal{P}_K^{n+1} \ | \ y \in x + \mathcal{P}_K^n} \ \phi_{K,n+1}(y + \mathcal{P}_K^{n+1}) \ &= |\mathcal{P}_K^n/\mathcal{P}_K^{n+1}||\mathcal{O}_K/\mathcal{P}_K|^{(1/2)-n-1} \\ &= |\mathcal{O}_K/\mathcal{P}_K|^{(1/2)-n}, \end{aligned}$$

as required. Usually the integer $|\mathcal{O}_K/\mathcal{P}_K|$ is denoted by q.

Now let $f \in C_c^{\infty}(K)$, the set of compactly supported and locally constant functions

$$f : K \longrightarrow \mathbb{C}.$$

This means (see §4.1) that there exists an integer $t \geq 0$, depending on f, such that

(i) $f(x) \neq 0$ implies that $x \in \mathcal{P}_K^{-t}$ and

(ii) if $x, y \in K$ and $x - y \in \mathcal{P}_K^t$ then $f(x) = f(y)$.

Recall that there is a chain of fractional ideals of the form

$$\mathcal{P}_K^{-r} \subset \mathcal{P}_K^{-r-1} \subset \ldots \ \subset K.$$

Choosing integers $r, n >> 0$ we define

$$I_K(f) = I_{K,r,n}(f)$$

which will serve as a formula for a Haar integral on K once we have verified invariance under right translation, which is seen as follows.

For $a \in K$ set $f_a(x) = f(a + x)$. Choose r so large that $a \in \mathcal{P}_K^{-r}$ then as $x + \mathcal{P}_K^n$ runs through $\mathcal{P}_K^{-r}/\mathcal{P}_K^n$ so does $a + x + \mathcal{P}_K^n$ and vice versa so that $I_{K,r,n}(f) = I_{K,r,n}(f_a)$, as required.

In the integral notation it is usual to write $I_K(f) = \int_K \ f(x)dx$, in the spirit of calculus!

Here is a second example of a measure. Let $\xi_n = e^{2\pi\sqrt{-1}/n} \in \mathbb{C}$. Take $K = \mathbb{Q}_p$ (p primes) and suppose that $0 < r << n$ are integers. For an integer a set $\Psi_{\mathbb{Q}_p,r,n}(\frac{a}{p^r}) = \xi_{p^r}^a$. The representatives of elements in $(p^{-r}\mathbb{Z}_p + p^n\mathbb{Z}_p)/p^n\mathbb{Z}_p$ are the fractions $\frac{a}{p^r}$ with $0 \leq a \leq p^{n+r-1}(p-1)$ and $HCF(a,p) = 1$. We have $\Psi_{\mathbb{Q}_p,r,n}((\frac{a}{p^r}) = \Psi_{\mathbb{Q}_p,r,n}((\frac{a+p^r}{p^r})$ so that each value on a representative is taken p^n times.

Define $\phi_{\mathbb{Q}_p,r,n}(\frac{a}{p^r}) = \frac{1}{p^n}\Psi_{\mathbb{Q}_p,r,n}((\frac{a}{p^r}))$. The representatives of $\frac{b+p^{n+r}}{p^r} \in \frac{b}{p^r} + p^{n+1}\mathbb{Z}_p$ which belong to $\frac{a}{p^r} + p^n\mathbb{Z}_p$ are

$$b = a, a+p^{n+r}, a+2p^{n+r}, \ldots, a+(p-1)p^{n+r}$$

so that

$$\sum_{\frac{b}{p^r}+p^{n+1}\mathbb{Z}_p \ =\frac{a}{p^r}+p^n\mathbb{Z}_p} \phi_{\mathbb{Q}_p,r,n+1}(\frac{b}{p^r}) = p\xi^a_{p^r}p^{-n-1} = \xi^a_{p^r}p^{-n} = \phi_{\mathbb{Q}_p,r,n}(\frac{a}{p^r}).$$

5. Example of GL_2K with K a non-Archimedean local field

Set $G_K = GL_2K$,

$$U = \{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \mid x \in K\} \cong K.$$

Take

$$H = Z(G_K)\cdot GL_2\mathcal{O}_K = \{ \begin{pmatrix} \pi_K^i a & \pi_K^i b \\ \pi_K^i c & \pi_K^i d \end{pmatrix} \mid i \in \mathbb{Z}, \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2\mathcal{O}_K\}.$$

Consider the representation $\pi = \mathrm{Ind}_H^{G_K}(\phi)$ where $\phi : H \longrightarrow \mathbb{C}^*$ is a continuous character which restrict to the central character $\underline{\phi} : K^* = Z(G_K) \longrightarrow \mathbb{C}^*$ which is common to all the representations under consideration. To examine π^U we consider the Double Coset Formula

$$\mathrm{Res}_U^{G_K}(\pi) = \oplus_{z\in U\backslash G_K/H} \ \mathrm{Ind}^U_{U\cap zHz^{-1}}((z^{-1})^*(\phi)).$$

We have a bijections

$$U\backslash G_K/H \leftrightarrow B/(B\bigcap H) \leftrightarrow (B/Z(G_K))/(B\bigcap H)/(Z(G_K)).$$

We have an isomorphism $(B/Z(G_K)) \cong K^* \times K$ which sends the coset of $\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix}$ to $(\alpha, \frac{\beta}{\alpha})$. The subgroup $(B\bigcap H)/(Z(G_K))$ equals $\mathcal{O}_K^* \times \mathcal{O}_K$ so that the set of double cosets $z \in U\backslash G_K/H$ is in a bijection with $\mathbb{Z}\times k$, where k is the residue field of K, and $(i, \underline{\beta})$ is represented by $\begin{pmatrix} \pi_K^i & \pi_K^i\underline{\beta} \\ 0 & 1 \end{pmatrix}$ where $\underline{\beta} \in \mathcal{O}_K$ runs through a choice of representatives for k. To determine

$U \bigcap zHz^{-1}$ consider

$$\begin{pmatrix} \pi_K^i & \pi_K^i \underline{\beta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \pi_K^j a & \pi_K^j b \\ \pi_K^j c & \pi_K^j d \end{pmatrix} \begin{pmatrix} \pi_K^{-i} & -\pi_K^{-i} \underline{\beta} \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \pi_K^{i+j} a + \pi_K^{i+j} c\underline{\beta} & \pi_K^{i+j} b + \pi_K^{i+j} d\underline{\beta} \\ \pi_K^j c & \pi_K^j d \end{pmatrix} \begin{pmatrix} \pi_K^{-i} & -\pi_K^{-i} \underline{\beta} \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \pi_K^j a + \pi_K^j c\underline{\beta} & -(\pi_K^j a + \pi_K^j c\underline{\beta})\underline{\beta} + \pi_K^{i+j} b + \pi_K^{i+j} d\underline{\beta} \\ \pi_K^{j-i} c & \pi_K^j d - \pi_K^{j-i} c\underline{\beta} \end{pmatrix}.$$

If $a, b, c, d \in \mathcal{O}_K$ and $\underline{\beta} = 0$ then $1 = \pi_K^j a$ so $j = 0$ and $a = 1$. Also $0 = \pi_K^i b$ so $b = 0$ and finally $d = 1$. In which case, for each integer i we have

$$\begin{pmatrix} \pi_K^i & 0 \\ 0 & 1 \end{pmatrix} H \begin{pmatrix} \pi_K^i & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \{ \begin{pmatrix} \pi_K^i & 0 \\ \pi_K^{-i} \cdot \mathcal{O}_K & 1 \end{pmatrix} \}$$

and

$$zHz^{-1} \bigcap U = \{ \begin{pmatrix} 1 & 0 \\ \mathcal{O}_K & 1 \end{pmatrix} \}.$$

If $a, b, c, d \in \mathcal{O}_K$ and $\underline{\beta} \neq 0$ then for this to belong to U we must have $j = 0$ and $a + c\underline{\beta} = 1 = d - \pi_K^{-i} c\underline{\beta}$ and

$$0 = -(\pi_K^j a + \pi_K^j c\underline{\beta})\underline{\beta} + \pi_K^{i+j} b + \pi_K^{i+j} d\underline{\beta} = -\underline{\beta} + \pi_K^i b + \pi_K^i d\underline{\beta}$$

so that $i = 0$ also and

$$zHz^{-1} \bigcap U = \{ \begin{pmatrix} 1 & 0 \\ \mathcal{O}_K & 1 \end{pmatrix} \}.$$

Therefore, for each $z \in U \backslash G_K / H$, we have a copy of $\mathrm{Ind}_{O_K}^K(\phi)$ where $U \cong K$ and $\dim(\mathrm{Ind}_{O_K}^K(\phi))^U = 1$ with a basis, in the functional model for induced representations, given by $f(w + u) = \phi(w) f(1)$ for $u \in K \cong U, w \in O_K$ - writing the group product as " $+$ " in K.

Note that it is important to take $\mathrm{Ind}_{\mathcal{O}_K}^K(\phi)$ rather than $c - \mathrm{Ind}_{\mathcal{O}_K}^K(\phi)$, since this function does not exist in the latter. The compact open subgroup K_f in condition (ii) of the functional definition of the induced representation can be taken as $\mathrm{Ker}(\phi) \bigcap U$ when the character is complex-valued, since $\mathbb{C}^*$ has the discrete topology.

The Double Coset Formula ([64] p.185) (in the tensor product notation for an induced representation) is an isomorphism of the form

$$\alpha : \mathrm{Res}_J^G \mathrm{Ind}_H^G(\phi) \stackrel{\cong}{\longrightarrow} \oplus_{z\in J\backslash G/H} \mathrm{Ind}_{J\cap zHz^{-1}}^J((z^{-1})^*(\phi))$$

given by the formula

$$\alpha(g \otimes_H w) = j \otimes_{J\cap zHz^{-1}} w \text{ if } g = jzh, j \in J, h \in H$$

and

$$\alpha^{-1}(j \otimes_{J\cap zHz^{-1}} w) = jz \otimes_H w \ \text{ if } j^{-1}g = zh, j \in J, h \in H.$$

Therefore if we take $J = U = \{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \mid x \in K\}$ and $H = K^*GL_2\mathcal{O}_K$ so that $H \bigcap zHz^{-1} = \{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \mid x \in \mathcal{O}_K\}$ for $z = \begin{pmatrix} \pi_K^i & \pi_K^i\underline{\beta} \\ 0 & 1 \end{pmatrix}$. Terefore

$$jz = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} \pi_K^i & \pi_K^i\underline{\beta} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \pi_K^i & \pi_K^i\underline{\beta} \\ x\pi_K^i & \pi_K^i\frac{1+\pi_K^i\underline{\beta}}{\pi_K^i} \end{pmatrix}.$$

In ([65] §12: Appendix: Comparison of Inductions) we find a dictionary for translation between the tensor product model for induced representations and the functional model. Taking $H = K^*GL_2\mathcal{O}_K$, $W = \mathbb{C}$ and $\phi : H \longrightarrow \mathbb{C}^*$ we find that $\mathrm{Ind}_{U\cap zHz^{-1}}^U((z^{-1})^*(\phi))$ corresponds to the space of functions $X_{(U\cap zHz^{-1},(z^{-1})^*(\phi))}$ $\{f_v, v \in \mathbb{C}\}$ such that

$$f_v(g) = \begin{cases} (z^{-1})^*(\phi)(g)v & \text{if } g \in U \cap zHz^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

If $g = jzh$ with $j \in U, h \in H$ then $\alpha^{-1}(j \cdot f_v) = jz \cdot f_v \in X_{(H,\phi)}$ where the latter is given by the formula for $(jz \cdot f_v) : H \longrightarrow \mathbb{C}$

$$(jz \cdot f_v)(g) = \begin{cases} \phi(gjz)v & \text{if } gjz \in H, \\ 0 & \text{otherwise.} \end{cases}$$

The formulae of ([65] §12: Appendix: Comparison of Inductions) show that this is a consistent system of relations and therefore that $\mathrm{Ind}_H^{GL_2K}(\phi)^U$ is one-dimensional in this example.

When the representations are of the complex numbers this example is presumably typical and $\mathrm{Ind}_H^{G_K}(\phi)^U$ is finite-dimensional[5].

[5] In ([65] §4 Extending the Definition of Admissibility) I give an extension of the definition of admissibility which applies in particular in the dipadic situation - i.e. K a p-adic local field and representations defined over the algebraic closure of K. It is certainly plausible that $\mathrm{Ind}_H^{G_K}(\phi)^U$ is finite-dimensional in this case, too. Currently I have left this investigation to the reader, since I may not have time to complete its analysis properly.

6. Appendix: Induced representations and locally profinite groups

Let G be a locally profinite group. In this section we are going to study admissible representations of G and its subgroups in relation to induction. These representations will be given by left-actions of the groups on vector spaces over k, which is an algebraically closed field of arbitrary characteristic.

Let us begin by recalling, from ([64] Chapter Two §1), induced and compactly induced smooth representations.

Definition 6.1.

Let G be a locally profinite group and $H \subseteq G$ a closed subgroup. Thus H is also locally profinite. Let

$$\sigma : H \longrightarrow \mathrm{Aut}_k(W)$$

be a smooth representation of H. Set X equal to the space of functions $f : G \longrightarrow W$ such that (writing simply $h \cdot w$ for $\sigma(h)(w)$ if $h \in H, w \in W$)

(i) $f(hg) = h \cdot f(g)$ for all $h \in H, g \in G$,

(ii) there is a compact open subgroup $K_f \subseteq G$ such that $f(gk) = f(g)$ for all $g \in G, k \in K_f$.

The (left) action of G on X is given by $(g \cdot f)(x) = f(xg)$ and

$$\Sigma : G \longrightarrow \mathrm{Aut}_k(X)$$

gives a smooth representation of G.

The representation Σ is called the representation of G smoothly induced from σ and is usually denoted by $\Sigma = \mathrm{Ind}_H^G(\sigma)$.

6.2.

$$(g \cdot f)(hg_1) = f(hg_1g) = hf(g_1g) = h(g \cdot f)(g_1)$$

so that $(g \cdot f)$ satisfies condition (i) of Definition 11.1.

Also, by the same discussion as in the finite group case (see Appndix §4), the formula will give a left G-representation, providing that $g \cdot f \in X$ when $f \in X$. However, condition (ii) asserts that there exists a compact open subgroup K_f such that $k \cdot f = f$ for all $k \in K_f$. The subgroup gK_fg^{-1} is also a compact open subgroup and, if $k \in K_f$, we have

$$(gkg^{-1}) \cdot (g \cdot f) = (gkg^{-1}g) \cdot f = (gk) \cdot f = (g \cdot (k \cdot f)) = (g \cdot f)$$

so that $g \cdot f \in X$, as required.

The smooth representations of G form an abelian category $\mathrm{Rep}(G)$.

Proposition 6.3.

The functor

$$\mathrm{Ind}_H^G : \mathrm{Rep}(H) \longrightarrow \mathrm{Rep}(G)$$

is additive and exact.

Proposition 6.4. *(Frobenius Reciprocity)*
There is an isomorphism

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\sigma)) \xrightarrow{\cong} \mathrm{Hom}_H(\pi, \sigma)$$

given by $\phi \mapsto \alpha \cdot \phi$ where α is the H-map

$$\mathrm{Ind}_H^G(\sigma) \longrightarrow \sigma$$

given by $\alpha(f) = f(1)$.

6.5. In general, if $H \subseteq Q$ are two closed subgroups there is a Q-map

$$\mathrm{Ind}_H^G(\sigma) \longrightarrow \mathrm{Ind}_H^Q(\sigma)$$

given by restriction of functions. Note that α in Proposition 11.4 is the special case where $H = Q$.

6.6. *The c-Ind variation*
Inside X let X_c denote the set of functions which are compactly supported modulo H. This means that the image of the support

$$\mathrm{supp}(f) = \{g \in G \mid f(g) \neq 0\}$$

has compact image in $H\backslash G$. Alternatively there is a compact subset $C \subseteq G$ such that $\mathrm{supp}(f) \subseteq H \cdot C$.

The Σ-action on X preserves X_c, since $\mathrm{supp}(g \cdot f) = \mathrm{supp}(f)g^{-1} \subseteq HCg^{-1}$, and we obtain $X_c = c - \mathrm{Ind}_H^G(W)$, the compact induction of W from H to G.

This construction is of particular interest when H is open. There is a canonical left H-map (see the Appendix in induction in the case of finite groups)

$$f : W \longrightarrow c - \mathrm{Ind}_H^G(W)$$

given by $w \mapsto f_w$ where f_w is supported in H and $f_w(h) = h \cdot w$ (so $f_w(g) = 0$ if $g \notin H$).

For $g \in G$ we have

$$(g \cdot f_w)(x) = f_w(xg) = \begin{cases} 0 & \text{if } xg \notin H, \\ (xg^{-1}) \cdot w & \text{if } xg \in H, \end{cases}$$

$$= \begin{cases} 0 & \text{if } x \notin Hg^{-1}, \\ (xg^{-1}) \cdot w & \text{if } x \in Hg^{-1}. \end{cases}$$

We shall be particularly interested in the case when $\dim_k(W) = 1$. In this case we write $W = k_\phi$ where $\phi : H \longrightarrow k^*$ is a continuous/smooth character and, as a vector space with a left H-action $W = k$ on which $h \in H$ acts by multiplication by $\phi(h)$. In this case α_c is an injective left $k[H]$-module homomorphism of the form

$$f : k_\phi \longrightarrow c - \mathrm{Ind}_H^G(k_\phi).$$

Lemma 6.7.

Let H be an open subgroup of G. Then

(i) $f : w \mapsto f_w$ is an H-isomorphism onto the space of functions $f \in c - \mathrm{Ind}_H^G(W)$ such that $\mathrm{supp}(f) \subseteq H$.

(ii) If $w \in W$ and $h \in H$ then $h \cdot f_w = f_{h \cdot w}$.

(iii) If $\mathcal{W}$ is a k-basis of W and $\mathcal{G}$ is a set of coset representatives for $H\backslash G$ then

$$\{g \cdot f_w \mid w \in \mathcal{W}, \ g \in \mathcal{G}\}$$

is a k-basis of $c - \mathrm{Ind}_H^G(W)$.

Proof

If $\mathrm{supp}(f)$ is compact modulo H there exists a compact subset C such that

$$\mathrm{supp}(f) \subseteq HC = \bigcup_{c \in C} Hc.$$

Each Hc is open so the open covering of C by the Hc's refines to a finite covering and so

$$C = Hc_1 \bigcup \ldots \bigcup Hc_n$$

and so

$$\mathrm{supp}(f) \subseteq HC = Hc_1 \bigcup \ldots \bigcup Hc_n.$$

For part (i), the map f is an H-homomorphism to the space of functions supported in H with inverse map $f \mapsto f(1)$.

For part (ii), from §6.6 we have

$$(h \cdot f_w)(x) = f_w(xh) = \begin{cases} 0 & \text{if } x \notin H, \\ \\ xh \cdot w & \text{if } x \in H. \end{cases}$$

so that, for all $x \in G$, $(h \cdot f_w)(x) = f_{h \cdot w}(x)$, as required.

For part (iii), the support of any $f \in c - \mathrm{Ind}_H^G(W)$ is a finite union of cosets Hg where the g's are chosen from the set of coset representatives $\mathcal{G}$ of $H\backslash G$. The restriction of f to any one of these Hg's also lies in $c - \mathrm{Ind}_H^G(W)$. If $\mathrm{supp}(f) \subseteq Hg$ then $(g \cdot f)(z) \neq 0$ implies that $zg \in Hg$ so that $g \cdot f$ has support contained in H. Hence $g \cdot f$ on H is a finite linear combination of the functions f_w with $w \in \mathcal{W}$. Therefore f is a finite linear combination of $g \cdot f_w$'s where $w \in \mathcal{W}, g \in \mathcal{G}$. Clearly the set of functions $g \cdot f_w$ with $g \in \mathcal{G}$ and $w \in \mathcal{W}$ is linearly independent. $\square$

Example 6.8. Let K be a p-adic local field with valuation ring $\mathcal{O}_K$ and π_K a generator of the maximal ideal of $\mathcal{O}_K$. Suppose that $G = GL_nK$ and that H is a subgroup containing the centre of G (that is, the scalar matrices K^*). If H is compact, open modulo K^* then there is a subgroup H' of finite index in H such that $H' = K^*H_1$ with H_1 compact, open in SL_nK. This can be established by studying the simplicial action of GL_nK on a suitable barycentric subdivision of the Bruhat-Tits building of $SL_n K$ (see [64] Chapter Four §1).

To show that H is both open and closed it suffices to verify this for H'. Firstly H' is open, since it is $H' = \bigcup_{z \in K^*} zH_1 = \bigcup_{s \in \mathbb{Z}} \pi_K^s H_1$.

Also $H' = K^* H_1$ is closed. Suppose that $X' \notin K^* H_1$. $K^* H_1$ is closed under mutiplication by the multiplicative group generated by π_K so that $\pi_K^m X' \notin K^* H_1$ for all m. By conjugation we may assume that H_1 is a subgroup of $SL_n \mathcal{O}_K$, which is the maximal compact open subgroup of $SL_n K$, unique up to conjugacy. Choose the smallest non-negative integer m such that every entry of $X = \pi_K^m X'$ lies in $\mathcal{O}_K$. Therefore we may write $0 \neq \det(X) = \pi_K^s u$ where $u \in \mathcal{O}_K^*$ and $1 \leq s$. Now suppose that V is an $n \times n$ matrix with entries in $\mathcal{O}_K$ such that $X + \pi_K^t V \in K^* H_1$. Then

$$\det(X + \pi_K^t V) \equiv \pi_K^s u \text{ (modulo } \pi_K^t).$$

So that if $t > s$ then s must have the form $s = nw$ for some integer w and $\pi_K^{-w}(X + \pi_K^t V) \in GL_n \mathcal{O}_K \bigcap K^* H_1 = H_1$. Therefore all the entries in $\pi_K^{-w} X$ lie in $\mathcal{O}_K$ and $\pi_K^{-w} X \in GL_n \mathcal{O}_K$. Enlarging t, if necessary, we can ensure that $\pi_K^{-w} X \in H_1$, since H_1 is closed (being compact), and therefore $X' \in K^* H_1$, which is a contradiction.

Since H is both closed and open in $GL_n K$ we may form the admissible representation $c - \mathrm{Ind}_H^{GL_n K}(k_\phi)$ for any continuous character $\phi : H \longrightarrow k^*$ and apply Lemma **??**.

If $g \in GL_n K, h \in H$ then $(g \cdot f_1)(x) = \phi(xg)$ if $xg \in H$ and zero otherwise. On the other hand, $(gh \cdot f_1)(x) = \phi(xgh) = \phi(h)\phi(xg)$ if $xg \in H$ and zero otherwise. Therefore as a left $GL_n K$-representation $c - \mathrm{Ind}_H^{GL_n K}(k_\phi)$ is isomorphic to

$$k[GL_n K]/(\phi(h)g - gh \mid g \in GL_n K,\ h \in H)$$

with left action induced by $g_1 \cdot g = g_1 g$.

This vector space is isomorphic to the k-vector space whose basis is given by k-bilinear tensors over H of the form $g \otimes_{k[H]} 1$ as in the case of finite groups. The basis vector $g \cdot f_1$ corresponds to $g \otimes_H 1$ and $GL_n K$ acts on the tensors by left multiplication, as usual (see Appendix §4 in the finite group case).

Proposition 6.9.

The functor

$$c - \mathrm{Ind}_H^G : \mathrm{Rep}(H) \longrightarrow \mathrm{Rep}(G)$$

is additive and exact.

Proposition 6.10.
Let $H \subseteq G$ be an open subgroup and (σ, W) smooth. Then there is a functorial isomorphism

$$\mathrm{Hom}_G(c - \mathrm{Ind}_H^G(W), \pi) \xrightarrow{\cong} \mathrm{Hom}_H(W, \pi)$$

given by $F \mapsto F \cdot f$, the composition with the H-map f of Lemma 6.7.

Example 6.11. $c - \underline{\mathrm{Ind}}_H^G(\phi)$
Suppose that $\phi : H \longrightarrow k^*$ is a continuous character (i.e. a one-dimensional smooth representation of H).

Suppose that we are in a situation analogous to that of Example 6.8. Namely suppose that H is open and closed, contains $Z(G)$,the centre of G, and is compact open modulo $Z(G)$. A basis for k is given by $1 \in k^*$ and we have the function $f_1 \in X_c$ given by $f_1(h) = \phi(h)$ if $h \in H$ and $f_1(g) = 0$ if $g \notin H$.

If, following Lemma 6.7, $\mathcal{G}$ is a set of coset representatives for $H\backslash G$ then a k-basis for $c - \underline{\mathrm{Ind}}_H^G(\phi)$ is given by

$$\{g \cdot f_1 \mid g \in \mathcal{G}\}.$$

For $g \in G$ we have

$$(g \cdot f_1)(x) = f_1(xg) \quad = \begin{cases} 0 & \text{if } xg \notin H, \\ \\ \phi(xg) & \text{if } xg \in H, \end{cases}$$

$$= \begin{cases} 0 & \text{if } x \notin Hg^{-1}, \\ \\ \phi(xg) & \text{if } x \in Hg^{-1}. \end{cases}$$

Before going further let us introduce the presence of (H, ϕ) into the notation.

Definition 6.12. Let H be a closed subgroup of G containing the centre, $Z(G)$, which is compact open modulo $Z(G)$. Let $\phi : H \longrightarrow k^*$ be a continuous character of H. Denote by $X_c(H, \phi)$ the k-vector space of functions $f : G \longrightarrow k$ such that

(i) $f(hg) = \phi(h)f(g)$ for all $h \in H, g \in G$,

(ii) there is a compact open subgroup $K_f \subseteq G$ such that $f(gk) = f(g)$ for all $g \in G, k \in K_f$,

(ii) f is compactly supported modulo H.

As in §6.6, the left action of G on $X_c(H, \phi)$ is given by $(g \cdot f)(x) = f(xg)$ and therefore

$$\Sigma : G \longrightarrow \mathrm{Aut}_k(X_c(H, \phi))$$

gives a smooth representation of G - denoted by $\Sigma = c - \mathrm{Ind}_H^G(\phi)$.

Henceforth we shall denote the map written as f_1 in Example 6.11 by $f_{(H,\phi)} \in X_c(H, \phi)$.

Therefore, for $g \in G$, we have

$$(g \cdot f_{(H,\phi)})(x) = f_{(H,\phi)}(xg) = \begin{cases} 0 & \text{if } xg \notin H, \\ \phi(xg) & \text{if } xg \in H, \end{cases}$$

$$= \begin{cases} 0 & \text{if } x \notin Hg^{-1}, \\ \phi(xg) & \text{if } x \in Hg^{-1}. \end{cases}$$

Definition 6.13. For (H, ϕ) and (K, ψ) as in Definition 6.12, write $[(K, \psi), g, (H, \phi)]$ for any triple consisting of $g \in G$, characters ϕ, ψ on subgroups $H, K \leq G$, respectively such that

$$(K, \psi) \leq (g^{-1}Hg, (g)^*(\phi))$$

which means that $K \leq g^{-1}Hg$ and that $\psi(k) = \phi(h)$ where $k = g^{-1}hg$ for $h \in H, k \in K$.

Let $\mathcal{H}$ denote the k-vector space with basis given by these triples. Define a product on these triples by the formula

$$[(H, \phi), g_1, (J, \mu)] \cdot [(K, \psi), g_2, (H, \phi)] = [(K, \psi), g_1g_2, (J, \mu)]$$

and zero otherwise. This product makes sense because
(i) if $K \leq g_2^{-1}Hg_2$ and $H \leq g_1^{-1}Jg_1$ then $K \leq g_2^{-1}Hg_2 \leq g_2^{-1}g_1^{-1}Jg_1g_2$ and
(ii) if $\psi(k) = \phi(h) = \mu(j)$, where $k = g_2^{-1}hg_2, h = g_1^{-1}jg_1$ then $k = g_2^{-1}g_1^{-1}jg_1g_2$.

This product is clearly associative and we define an algebra $\mathcal{H}_{cmc}(G)$ to be $\mathcal{H}$ modulo the relations (c.f. Appendix §4)

$$[(K, \psi), gk, (H, \phi)] = \psi(k^{-1})[(K, \psi), g, (H, \phi)]$$

and

$$[(K, \psi), hg, (H, \phi)] = \phi(h^{-1})[(K, \psi), g, (H, \phi)].$$

We observe that

$$[(K, \psi), g, (H, \phi)] = [(g^{-1}Hg, g^*\phi), g, (H, \phi)] \cdot [(K, \psi), 1, (g^{-1}Hg, g^*\phi)]$$

We shall refer to this algebra as the compactly supported modulo the centre (CSMC-algebra) of G.

Lemma 6.14.
Let $[(K, \psi), g, (H, \phi)]$ be a triple as in Definition 6.13. Associated to this triple define a left $k[G]$-homomorphism

$$[(K, \psi), g, (H, \phi)] : X_c(K, \psi) \longrightarrow X_c(H, \phi)$$

by the formula $g_1 \cdot f_{(K,\psi)} \mapsto (g_1g^{-1}) \cdot f_{(H,\phi)}$.

For a proof, which is the same as in the case when G is finite, can be found in (the Appendix on induction in the case of finite groups).

Theorem 6.15.

Let $\mathcal{M}_c(G)$ denote the partially order set of pairs (H, ϕ) as in Definitions 6.12 and 6.13 (so that $X_c(H, \phi) = c - \mathrm{Ind}_H^G(\phi)$). Then, when each $n_\alpha = 1$,

$$M_c(\underline{n}, G) = \oplus_{\alpha \in \mathcal{A}, (H,\phi) \in \mathcal{M}_c(G)} \ n_\alpha X_c(H, \phi)$$

is a left $k[G] \times \mathcal{H}_{cmc}(G)$-module. For a general distribution of multiplicities $\{n_\alpha\}$ it is Morita equivalent to a left $k[G] \times \mathcal{H}_{cmc}(G)$-module.

Proof

We have only to verify associativity of the module multiplication, which is obvious. □

Definition 6.16. ${}_{k[G]}$**mon**, *the monomial category of* G

The monomial category of G is the additive category (non-abelian) whose objects are the k-vector spaces given by direct sums of $X_c(H, \phi)$'s of §6.15 and whose morphisms are elements of the hyperHecke algebra $\mathcal{H}_{cmc}(G)$. In other words the subcategory of the category of $k[G] \times \mathcal{H}_{cmc}(G)$-modules of which one example is $M_c(\underline{n}, G)$ in §6.15.

7. Appendix: On p-adic Artin L-functions

This paper has been include (occupying §§6-10) i order to recall thsi proof of inductivity and naturality of the classical Artin L-functions, since the samr discussion should apply to the 2-variable L-function as constructed for induced representations in §2.

In this paper I want to introduce a conjecture (Conjecture 11.2) whose cryptic form would be: "The Wiles Unit is a determinant". The "Wiles Unit" unit is the p-adic unit-valued function on Galois representations given by the ratio of the p-adic L-function to the Iwasawa polynomial. The values of this function are p-adic units by the main result of [71]. In §2 I give a functorial treatment of the material of [34], which results in Iwasawa polynomials even in the non-abelian case. §3 recalls the definition of the p-adic L-function. In §4 we examine what a determinantal functions are and how to detect them. Having set up the background material we are ready in §5 to state the conjecture and to accompany it with some evidence in its favour.

This conjecture grew out of my collaboration with Ted Chinburg, Manfred Kolster and Georges Pappas. In a future paper we shall prove some non-trivial abelian cases.

8. The Main Construction

8.1. Let E/F be a Galois extension of totally real number fields with Galois group, $G(E/F)$. Let p be a prime and let $\mathbf{Z}_p$ denote the p-adic integers. Let $\overline{\mathbf{Q}}_p$ be an algebraic closure of $\mathbf{Q}_p$, the p-adics. Let F_∞/F be the cyclotomic $\mathbf{Z}_p$-extension so that $\Gamma = G(F_\infty/F) \cong \mathbf{Z}_p$ and we choose, once and for all, a topological generator, γ, for Γ.

Let $X_{E/F}$ denote the Galois group, $G(L/EF_{\infty})$, where L/EF_{∞} is the maximal abelian pro-p extension in which only primes over p ramify. Hence $X_{E/F}$ is a module over the completed (with respect to the profinite topology on the Galois group) group-ring , $\mathbb{Z}_p[G(EF_{\infty}/F)]$, if $g \in G(EF_{\infty}/F)$ acts on $x \in X_{E/F}$ by $g(x) = \overline{g}x\overline{g}^{-1}$ for any lifting of g to an F-automorphism, $\overline{g}$, of L.

Now let $\mathcal{O}_K$ denote the ring of integers in a p-adic local field, $K \subset \overline{\mathbb{Q}}_p$. Suppose that U is an $\mathcal{O}_K[G(EF_{\infty}/F)]$-module which is free of finite rank as an $\mathcal{O}_K$-module. A primary source of such U's, by inflation, is the set of $\mathcal{O}_K[G(E/F)]$-lattices of finite rank. Let

$$Hom(U, X_{E/F} \otimes_{\mathbb{Z}_p} \mathcal{O}_K)$$

denote the set of $\mathcal{O}_K$-module homomorphisms, $U \longrightarrow X_{E/F} \otimes_{\mathbf{Z}_p} \mathcal{O}_K$, endowed with the (left) $\mathcal{O}_K[G(EF_{\infty}/F)]$-module structure given by $h(f)(u) = h(f(h^{-1}(u)))$ for $h \in G(EF_{\infty}/F)$. Since $X_{E/F}$ is a Noetherian $\mathcal{O}_K[G(EF_{\infty}/F)]$-module so is $Hom(U, X_{E/F} \otimes_{\mathbf{Z}_p} \mathcal{O}_K)$.

The $G(EF_{\infty}/F_{\infty})$-fixed points of $Hom(U, X_{E/F} \otimes_{\mathbf{Z}_p} \mathcal{O}_K)$, written

$$I_{E/F}(U) = Hom_{G(EF_{\infty}/F_{\infty})}(U, X_{E/F} \otimes_{\mathbf{Z}_p} \mathcal{O}_K),$$

is a Noetherian, torsion module over $\mathcal{O}_K[\Gamma] \cong \mathcal{O}_K[[T]]$ if $T+1$ acts like $\gamma \in G(EF_{\infty}/F)/G(EF_{\infty}/F_{\infty}) \cong \Gamma$. The $\overline{\mathbb{Q}}_p$-vector space, $V_{E/F} = X_{E/F} \otimes_{\mathbb{Z}_p} \overline{\mathbf{Q}}_p$, is finite-dimensional and so is

$$I_{E/F}(U) \otimes_{\mathcal{O}_K} \overline{\mathbb{Q}}_p \cong Hom_{G(EF_{\infty}/F_{\infty})}(U \otimes_{\mathcal{O}_K} \overline{\mathbb{Q}}_p, X_{E/F} \otimes_{\mathbf{Z}_p} \overline{\mathbf{Q}}_p).$$

Therefore the characteristic polynomial, $det(tI - ((\gamma - 1) \cdot -))$, of $\gamma - 1$ acting on $I_{E/F}(U) \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p$ is a monic polynomial with coefficients in $\mathcal{O}_K$. This characteristic polynomial clearly depends only on the $\overline{\mathbb{Q}}_p$-representation

$$\rho : G(EF_{\infty}/F) \longrightarrow Aut_{\overline{\mathbb{Q}}_p}(U \otimes_{\mathcal{O}_K} \overline{\mathbb{Q}}_p).$$

Hence we set

$$h_{\rho}(T) = det(T - ((\gamma - 1) \cdot -) \mid I_{E/F}(U) \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p) \in \mathcal{O}_K[T].$$

Example 8.2. Suppose that that $E \cap F_{\infty} = F$ and that ρ is one-dimensional, coming from a homomorphism of the form $\rho : G(E/F) \longrightarrow \mathcal{O}_K^*$. We have a commutative diagram of canonical maps of Galois groups.

$$\begin{tikzcd}
 & G(EF_\infty/E) \arrow[d, "i"] \arrow[dr, "\cong"] & \\
G(EF_\infty/F_\infty) \arrow[r, "i'"] \arrow[dr, "\cong"'] & G(EF_\infty/F) \arrow[r, "\pi'"] \arrow[d, "\pi"] & \Gamma \\
 & G(E/F) &
\end{tikzcd}$$

Let $\gamma' \in G(EF_\infty/E)$ denote the unique element which satisfies $\pi'(i(\gamma')) = \gamma$ and set $\psi = \rho \cdot \pi : G(EF_\infty/F) \longrightarrow \overline{\mathbb{Q}}_p^*$. Then ψ is a one-dimensional representation of type S in the terminology of [34]. That is, the fixed field of the kernel of ψ is linearly disjoint from F_∞ and ψ annihilates some lift of γ (for example, γ').

There is an isomorphism of $\mathcal{O}_K[\Gamma]$-modules of the form

$$\lambda : I_{E/F}(U) \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p \cong Hom_{G(EF_\infty/F_\infty)}(U \otimes_{\mathcal{O}_K} \overline{\mathbb{Q}}_p, V_{E/F}) \longrightarrow V_{E/F}^{\psi}$$

where

$$V_{E/F}^{\psi} = \{v \in V_{E/F} \mid g(v) = \psi(g)v \text{ for all } g \in G(EF_\infty/F_\infty) \cong G(E/F)\}.$$

In fact, $\lambda(f) = f(1)$ is such an isomorphism, since $g(\lambda(f)) = f(g(1)) = f(\psi(1)) = \psi(1)f(1) = \psi(1)\lambda(f)$.

This means that $h_\rho(T)$ defined in §8.1 coincides with the $h_\rho(T)$ (or, equivalently, with $h_\psi(T)$) of ([34] §1).

In the terminology of [34] a character of type W is a one-dimensional representation of the form

$$\rho' : G(EF_\infty/F) \xrightarrow{\pi'} \Gamma \longrightarrow \overline{\mathbb{Q}}_p^*.$$

In ([34] Proposition 3) the process of twisting representations by characters of type W to obtain representations of type S is studied in relation to its effect on $h_\psi(T)$.

The following result describes the effect of twisting by by characters of type W in general.

Proposition 8.3.
Let

$$\rho' : G(EF_\infty/F) \xrightarrow{\pi'} G(EF_\infty/F)/G(EF_\infty/F_\infty) \cong \Gamma \xrightarrow{\overline{\rho}} \overline{\mathbb{Q}}_p^*$$

be a one-dimensional representation of type W in the terminology of [34]. *Then, in the notation of §8.1,*

$$h_{\rho\rho'}(T) = \overline{\rho}(\gamma)^{-d} h_\rho(\overline{\rho}(\gamma)(1+T)-1)$$

where $d = dim_{\overline{\mathbf{Q}}_p}(I_{E/F}(U(\rho)) \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p)$.

Proof
Let $U(\rho)$ and $U(\rho\rho')$ be the modules associated with ρ and $\rho\rho'$ respectively. The action of $\gamma \in \Gamma$ on $f \in I_{E/F}(U(\rho)) \otimes_{\mathcal{O}_K} \overline{\mathbb{Q}}_p$ is given by lifting γ to $\gamma' \in G(EF_\infty/F)$, and conjugating $\gamma(f)(-) = \gamma'(f((\gamma')^{-1}(-)))$. If we identify the vector spaces, $U(\rho\rho') \otimes_{\mathcal{O}_K} \overline{\mathbb{Q}}_p$ and $U(\rho) \otimes_{\mathcal{O}_K} \overline{\mathbb{Q}}_p$, then the action of γ' on the former is equal to $\rho'(\gamma') = \overline{\rho}(\gamma)$ times the action of γ' on the latter. Therefore, if A is a matrix which describes the action of γ on $I_{E/F}(U(\rho)) \otimes_{\mathcal{O}_K} \overline{\mathbb{Q}}_p$, with respect to some choice of basis, then $\overline{\rho}(\gamma)^{-1}A$ is the corresponding matrix on $I_{E/F}(U(\rho\rho')) \otimes_{\mathcal{O}_K} \overline{\mathbb{Q}}_p$. Computing characteristic polynomials of $\gamma - 1$, we obtain

$$\begin{aligned} h_{\rho\rho'}(T) &= det(T + 1 - \overline{\rho}(\gamma)^{-1}A) \\ &= \overline{\rho}(\gamma)^{-d} det(\overline{\rho}(\gamma)(1+T) - A) \\ &= \overline{\rho}(\gamma)^{-d} det((\overline{\rho}(\gamma)(1+T) - 1) - A + 1) \\ &= \overline{\rho}(\gamma)^{-d} h_\rho(\overline{\rho}(\gamma)(1+T) - 1), \end{aligned}$$

as required. □

Proposition 8.4.
Let N/F be an intermediate Galois extension of E/F and let U be an $\mathcal{O}_K[G(NF_\infty/F)]$-module which is free of finite rank as an $\mathcal{O}_K$-module. Then there is an isomorphism of Γ-representations of the form

$$I_{E/F}(Inf_{G(NF_\infty/F)}^{G(EF_\infty/F)}(U)) \otimes_{\mathcal{O}_K} \overline{\mathbb{Q}}_p \xrightarrow{\cong} I_{N/F}(U) \otimes_{\mathcal{O}_K} \overline{\mathbb{Q}}_p,$$

where Inf denotes inflation.

Proof
Since $G(EF_\infty/NF_\infty)$ acts trivially on $Inf_{G(NF_\infty/F)}^{G(EF_\infty/F)}(U)$, we have an isomorphism

$$\begin{aligned} &Hom_{G(EF_\infty/F_\infty)}(Inf_{G(NF_\infty/F)}^{G(EF_\infty/F)}(U) \otimes_{\mathcal{O}_K} \overline{\mathbb{Q}}_p, V_{E/F}) \\ &\cong Hom_{G(NF_\infty/F_\infty)}(U \otimes_{\mathcal{O}_K} \overline{\mathbf{Q}}_p, V_{E/F}^{G(EF_\infty/NF_\infty)}). \end{aligned}$$

However the natural map, $X_{E/F} \longrightarrow X_{N/F}$, induces an isomorphism

$$(V_{E/F})_{G(EF_\infty/NF_\infty)} \xrightarrow{\cong} V_{N/F},$$

where $(A)_G$ denotes the coinvariants of G acting on A. Also the norm, $N(v) = \sum_{g \in G(EF_\infty/NF_\infty)} g(v)$, induces an isomorphism

$$N : (V_{E/F})_{G(EF_\infty/NF_\infty)} \xrightarrow{\cong} (V_{E/F})^{G(EF_\infty/NF_\infty)}.$$

Since a lifting of $\gamma \in \Gamma$ to $G(EF_\infty/F)$ maps to a lifting of γ to $G(NF_\infty/F)$, it is clear that the resulting isomorphism commutes with the Γ-action, as required. $\square$

Proposition 8.5.
Let N/F be an intermediate extension of E/F and let U be an $\mathcal{O}_K[G(EF_\infty/N)]$-module which is free of finite rank as an $\mathcal{O}_K$-module. Then there is an isomorphism of $\mathcal{O}_K[\Gamma]$-modules of the form

$$I_{E/F}(Ind_{G(EF_\infty/N)}^{G(EF_\infty/F)}(U)) \xrightarrow{\cong} Ind_{G(N_\infty/N)}^{\Gamma}(I_{E/N}(U)),$$

where Ind denotes induction. Here $G(N_\infty/N)$ has been identified with $G(F_\infty/N \cap F_\infty) \subseteq \Gamma$.

Proof
We have $F \subseteq N \cap F_\infty \subseteq N \subseteq E$ and $G(N \cap F_\infty/F) \cong \mathbf{Z}/p^m$, generated by γ, since it is a finite quotient of $\mathbf{Z}_p$. Therefore $1, \gamma, \ldots, \gamma^{p^m-1}$ constitute a set of representatives of the double cosets

$$G(EF_\infty/F_\infty)\backslash G(EF_\infty/F)/G(EF_\infty/N) \equiv \Gamma/G(F_\infty/N \cap F_\infty).$$

By the Double Coset Formula [62], $Ind_{G(EF_\infty/N)}^{G(EF_\infty/F)}(U)$ is equal as an $\mathcal{O}_K[G(EF_\infty/F_\infty)]$-module to

$$Res_{G(EF_\infty/F_\infty)}^{G(EF_\infty/F)}(Ind_{G(EF_\infty/N)}^{G(EF_\infty/F)}(U))$$

$$\cong \oplus_{0 \leq i \leq p^m-1} Ind_{G(EF_\infty/F_\infty) \cap \gamma^i G(EF_\infty/N)\gamma^{-i}}^{G(EF_\infty/F_\infty)}((\gamma^{-i})^*(U))$$

where $(\gamma^{-i})^*(U)$ is equal to a copy of U on which $\gamma^i g \gamma^{-i}$ acts as g does on U. Hence

$$Res_{G(EF_\infty/F_\infty)}^{G(EF_\infty/F)}(Ind_{G(EF_\infty/N)}^{G(EF_\infty/F)}(U)) \cong \oplus_{0 \leq i \leq p^m-1} Ind_{G(EF_\infty/\gamma^i(N)F_\infty)}^{G(EF_\infty/F_\infty)}((\gamma^{-i})^*(U)).$$

It is sufficient to prove the proposition in the two special cases (a) $N \cap F_\infty = F$ and (b) $N \subset F_\infty$ because then we have a chain of isomorphisms of the following form:

$$\begin{aligned} I_{E/F}(Ind^{G(EF_\infty/F)}_{G(EF_\infty/N)}(U)) &\cong I_{E/F}(Ind^{G(EF_\infty/F)}_{G(EF_\infty/N\cap F_\infty)}(Ind^{G(EF_\infty/N\cap F_\infty)}_{G(EF_\infty/N)}(U))) \\ &\cong Ind^{\Gamma}_{G(F_\infty/N\cap F_\infty)}(I_{E/N\cap F_\infty}(Ind^{G(EF_\infty/N\cap F_\infty)}_{G(EF_\infty/N)}(U))) \\ &\cong Ind^{\Gamma}_{G(F_\infty/N\cap F_\infty)}(Ind^{G(F_\infty/N\cap F_\infty)}_{G(F_\infty/N\cap F_\infty)}(I_{E/N}(U))) \\ &\cong Ind^{\Gamma}_{G(F_\infty/N\cap F_\infty)}(I_{E/N}(U)). \end{aligned}$$

In case (a) we have $m = 0$ and $N_\infty = NF_\infty$ so that $EN_\infty = EF_\infty$ and $X_{E/F} = X_{E/N}$. Hence, in this case,

$$Hom_{G(EF_\infty/F_\infty)}(Ind^{G(EF_\infty/F)}_{G(EF_\infty/N)}(U), X_{E/F} \otimes_{\mathbf{Z}_p} \mathcal{O}_K)$$

$$\cong Hom_{G(EF_\infty/F_\infty)}(Ind^{G(EF_\infty/F_\infty)}_{G(EF_\infty/F_\infty N)}(U), X_{E/F} \otimes_{\mathbf{Z}_p} \mathcal{O}_K)$$

$$\cong Hom_{G(EF_\infty/N_\infty)}(U, X_{E/N} \otimes_{\mathbf{Z}_p} \mathcal{O}_K),$$

as required.

In case (b) $N_\infty = F_\infty$, $X_{E/F} = X_{E/N}$ and $\gamma^i(N) \subset \gamma^i(F_\infty) = F_\infty$ so that

$$Hom_{G(EF_\infty/F_\infty)}(Ind^{G(EF_\infty/F)}_{G(EF_\infty/N)}(U), X_{E/F} \otimes_{\mathbf{Z}_p} \mathcal{O}_K)$$

$$\cong \oplus_{0\leq i\leq p^m-1} Hom_{G(EF_\infty/F_\infty)}(Ind^{G(EF_\infty/F_\infty)}_{G(EF_\infty/\gamma^i(N)F_\infty)}((\gamma^{-i})^*(U)), X_{E/F} \otimes_{\mathbf{Z}_p} \mathcal{O}_K)$$

$$\cong \oplus_{0\leq i\leq p^m-1} Hom_{G(EN_\infty/N_\infty)}((\gamma^{-i})^*(U), X_{E/N} \otimes_{\mathbf{Z}_p} \mathcal{O}_K),$$

since $\gamma^i(N)F_\infty = F_\infty$. However, $1, \gamma, \ldots, \gamma^{p^m-1}$ are also coset representatives for $\Gamma/G(F_\infty/N)$ and it is easy to see that

$$\oplus_{0\leq i\leq p^m-1} Hom_{G(EN_\infty/N_\infty)}((\gamma^{-i})^*(U), X_{E/N} \otimes_{\mathbf{Z}_p} \mathcal{O}_K)$$

$$\cong Ind^{\Gamma}_{G(F_\infty/N)}(Hom_{G(EN_\infty/N_\infty)}(U, X_{E/N} \otimes_{\mathbf{Z}_p} \mathcal{O}_K))$$

as Γ-modules. □

9. p-adic L-functions

9.1. Let E/F and $\mathcal{O}_K$ be as in §8.1. Let p be an odd prime. Let ρ be a one-dimensional $\overline{\mathbf{Q}}_p$-representation which is realised over $\mathcal{O}_K$. That is, $\mathcal{O}_K = U$ is an $\mathcal{O}_K[G(EF_\infty/F)]$-module associated to $\rho : G(E/F) \longrightarrow \overline{\mathbb{Q}}_p^*$ as in §8.1. The existence of the p-adic L-function, $L_p(s, \rho)$, was first shown by Deligne and Ribet [31] (see also [4] [20] [51] [52]).

Let S_p denote the set of all primes of F, $P \lhd \mathcal{O}_F$, above p and let $\sigma : \overline{\mathbf{Q}}_p \xrightarrow{\cong} \mathbf{C}$ be a fixed isomorphism. Thus $\sigma(\rho \cdot \omega^{-n})$ is a one-dimensional complex Galois representation of F, where $\omega : G(F(\mu_p)/F) \longrightarrow \mu_{p-1} \subset \mathbf{Z}_p^*$ denotes the Teichmüller character, μ_n denoting the n-th roots of unity. The function, $L_p(s, \rho)$, is characterised as the unique continuous $\overline{\mathbf{Q}}_p$-valued function of $s \in \mathbf{Z}_p - \{1\}$ (even continuous at $s = 1$ if ρ is non-trivial) such that

$$L_p(1-n, \rho) = \sigma^{-1}(L_{F,S_p}(1-n, \sigma(\rho \cdot \omega^{-n})))$$

for all integers $n \geq 1$. Here $L_{F,S_p}(1-n, -)$ denotes the Artin L-function associated to F after the removal of the Euler factors attached the primes in S_p.

Let $F' = F(\mu_p)$ so that F'_∞ contains all p-primary roots of unity. The action of $\Gamma = G(F_\infty/F) \cong G(F'_\infty/F')$ on an arbitrary p-primary root of unity, ξ, is given by a homomorphism, $\kappa : \Gamma \longrightarrow \mathbf{Z}_p^*$, determined by the formula, $g(\xi) = \xi^{\kappa(g)}$ for all $g \in \Gamma$.

Set $u = \kappa(\gamma) \in \mathbf{Z}_p^*$ so that $u \equiv 1$ (modulo p^m) if $\mu_{p^m} \subset F'$. Let $\mathbf{Z}_p[\rho]$ denotes the $\mathbf{Z}_p$-algebra generated by the character-values of ρ. If ρ is one-dimensional, as above, there exists a power series

$$G_\rho(T) \in \mathbf{Z}_p[\rho][[T]]$$

such that, except for $s = 0$ when ρ is trivial,

$$L_p(1-s, \rho) = \frac{G_\rho(u^s - 1)}{H_\rho(u^s - 1)}$$

for $s \in \mathbf{Z}_p$. Here

$$H_\rho(T) = \begin{cases} \rho(\gamma)(1+T) - 1, & \textit{if } \rho \textit{ is type } W, \\ \\ 1 & \textit{otherwise.} \end{cases}$$

Also, as explained in [52], if ρ' is of type W then

$$G_{\rho\rho'}(T) = G_\rho(\rho'(\gamma)(1+T) - 1) \text{ and } H_{\rho\rho'}(T) = H_\rho(\rho'(\gamma)(1+T) - 1).$$

If $\pi \in \mathbf{Z}_p[\rho]$ is a uniformiser then, by the Weierstrass Preparation Theorem, we may write $G_\rho(T)$ uniquely in the form

$$G_\rho(T) = \pi^{\mu(\rho)} G_\rho^*(T) U_\rho(T)$$

where $G_\rho^*(T)$ is a distinguished polynomial and $U_\rho(T)$ is a unit in the power series ring, $\mathbf{Z}_p[\rho][[T]]$. A distinguished polynomial is a monic polynomial all of whose non-leading coefficients lie in $\pi\mathbf{Z}_p[\rho]$.

If ρ is one-dimensional and of type S then it is shown in ([71] Theorem 1.3) that the polynomial, $h_\rho(T)$, of §8.1 satisfies

$$h_\rho(T) = G_\rho^*(T) \in \mathbf{Z}_p[\rho][T].$$

Since any one-dimensional representation may be written as the product of one of type S with one of type W, the previous discussion together with Proposition 8.3 shows that the above relation holds for all one-dimensional ρ (the factor $\overline{\rho}(\gamma)^{-d}$ being subsumed into the unit, $U_\rho(T)$).

9.2. In the following it will be wise to keep track of the identity of the base-field, F, in $G(E/F)$. Accordingly, we shall temporarily elaborate upon the notation of §9.1 and write $L_{p,F}(1-s,\rho)$, $\kappa_F : G(F_\infty/F) \longrightarrow \mathbf{Z}_p^*$, $\gamma_F \in G(F_\infty/F)$ and $u_F = \kappa_F(\gamma_F)$ for $L_p(1-s,\rho)$, κ, γ and u, respectively. When $F \subseteq N \subseteq E$ we choose $\gamma_N = \gamma_F^{[N\cap F_\infty:F]}$, identifying $G(N_\infty/N)$ with a subgroup of Γ as in Proposition 8.5.

Now let ρ be an arbitrary finite-dimensional $\overline{\mathbf{Q}}_p$-representation of $G(E/F)$. For such ρ the additivity and the invariance under inflation and induction of the Artin L-function imply that

$$L_{p,F}(1-s,\rho_1\oplus\rho_2) = L_{p,F}(1-s,\rho_1)L_{p,F}(1-s,\rho_2),$$

$$L_{p,F}(1-s, Inf_{G(N/F)}^{G(E/F)}(\rho)) = L_{p,F}(1-s,\rho),$$

$$L_{p,F}(1-s, Ind_{G(E/N)}^{G(E/F)}(\rho)) = L_{p,N}(1-s,\rho)$$

for the appropriate ρ,ρ_1,ρ_2 and $F\subseteq N\subseteq E$.

By Brauer's Induction Theorem there exist integers, n_i, and one-dimensional representations of subgroups, $\rho_i : G(E/N_i) \longrightarrow \overline{\mathbf{Q}}_p^*$ such that

$$\rho = \sum_i n_i Ind_{G(E/N_i)}^{G(E/F)}(\rho_i)$$

in the representation ring of $G(E/F)$. In fact, there is a canonical form for this induction formula which we may use if need be ([62] Theorem 2.3.9) in which each pair, $(G(E/N_i),\rho_i)$, is unique up to $G(E/F)$-conjugacy. Note that some of the n_i's may be negative integers. Therefore we obtain

$$\begin{aligned} L_{p,F}(1-s,\rho) &= \prod_i \ L_{p,N_i}(1-s,\rho_i)^{n_i} \\ &= \prod_i \ \Big(\frac{\pi(\rho_i)^{\mu(\rho_i)}h_{\rho_i}(u_{N_i}^s-1)U_{\rho_i}(u_{N_i}^s-1)}{H_{\rho_i}(u_{N_i}^s-1)}\Big)^{n_i} \end{aligned}$$

where $\pi(\rho_i)\in\mathbf{Z}_p[\rho_i]$ is a uniformiser.

The following result is mentioned, without proof, in ([34] p.82):

Proposition 9.3.
If ρ is irreducible and $dim_{\overline{\mathbf{Q}}_p}(\rho)\geq 2$ in §9.2 then

$$\prod_i H_{\rho_i}(u_{N_i}^s-1)^{n_i} = \pm 1.$$

Proof

The appearance in the denominator of the formula for $L_{p,F}(1-s,\rho)$ of $H_{\rho_i}(u_{N_i}^s-1)$ with $H_{\rho_i}(T)$ non-trivial occurs only when $\rho_i : G(E/N_i) \longrightarrow \overline{\mathbf{Q}}_p^*$ is of type W. This is equivalent to the restriction,

$$(\rho_i \mid G(E/E\cap F_\infty)\cap G(E/N_i) = G(E/(E\cap F_\infty)N_i),$$

being trivial.

Since $G(E/E \cap F_\infty) \lhd G(E/F)$ we may apply the operation of taking $G(E/E \cap F_\infty)$-coinvariants (which is the same as taking $G(E/E \cap F_\infty)$-fixed points) to both sides of the equation for ρ (see [**?**] Exercises 2.5.13-2.5.15).

The $G(E/E \cap F_\infty)$-fixed points of ρ form a subrepresentation which must either be trivial or all of ρ, since ρ is irreducible. However, if ρ were trivial on $G(E/E \cap F_\infty)$ it would factorise through the cyclic quotient, $G(E \cap F_\infty/F)$, and would therefore be one-dimensional. Therefore we have the relation

$$0 = \sum_i n_i (Ind_{G(E/N_i)}^{G(E/F)}(\rho_i))^{G(E/E\cap F_\infty)} = \sum_i n_i Ind_{G(E/(N_i \cap F_\infty))}^{G(E/F)}(\hat{\rho}_i)$$

where $\hat{\rho}_i$ on $G(E/(N_i \cap F_\infty)) = G(E/E \cap F_\infty))G(E/N_i)$ is given by $\hat{\rho}_i(xy) = \rho_i(y)$ for $x \in G(E/E \cap F_\infty))$ and $y \in G(E/N_i)$. These representations are all inflated from the cyclic quotient group, $G(E \cap F_\infty/F) \cong \mathbf{Z}/p^m$ so that in the representation ring of $G(E \cap F_\infty/F)$ we have the relation

$$0 = \sum_i n_i Ind_{G((E\cap F_\infty)/(N_i \cap F_\infty))}^{G((E\cap F_\infty)/F)}(\hat{\rho}_i).$$

Let $char_i(t)$ denote the characteristic polynomial

$$char_i(t) = det(tI - \gamma \mid Ind_{G((E\cap F_\infty)/(N_i \cap F_\infty))}^{G((E\cap F_\infty)/F)}(\hat{\rho}_i)) \in \overline{\mathbf{Q}}_p[t],$$

where γ is the image of $\gamma \in \Gamma$ in $G((E \cap F_\infty)/F) \cong \mathbf{Z}/p^m$. Hence we have the relation

$$1 = \prod_i char_i(u_F^{-s})^{n_i}.$$

However, using the base $1 \otimes 1, \gamma \otimes 1, \ldots, \gamma^{[N_i \cap F_\infty : F]-1} \otimes 1$ for $Ind_{G((E\cap F_\infty)/(N_i \cap F_\infty))}^{G((E\cap F_\infty)/F)}(\hat{\rho}_i))$ one sees easily that

$$char_i(t) = det \begin{pmatrix} t & 0 & 0 & \ldots & \ldots & 0 & -\rho_i(\gamma_F^{[N_i \cap F_\infty : F]}) \\ -1 & t & 0 & \ldots & \ldots & 0 & 0 \\ 0 & -1 & t & \ldots & \ldots & 0 & 0 \\ 0 & 0 & -1 & t & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \ldots & -1 & t \end{pmatrix}$$

$$= (t^{[N_i \cap F_\infty : F]} - \rho_i(\gamma_F^{[N_i \cap F_\infty : F]})).$$

However, $\gamma_F^{[N_i \cap F_\infty : F]} = \gamma_{N_i}$ and $u_F^{[N_i \cap F_\infty : F]} = u_{N_i}$ so that the relation becomes

$$\begin{aligned} 1 &= \prod_i \; (u_{N_i}^{-s} - \rho_i(\gamma_{N_i}))^{n_i} \\ &= (\prod_i \; u_{N_i}^{-sn_i})(\prod_i \; (1 - \rho_i(\gamma_{N_i}) u_{N_i}^s)^{n_i}) \\ &= (u_F^{-s})^{\sum_i n_i [N_i \cap F_\infty : F]} \prod_i \; (-H_{\rho_i}(u_{N_i}^s - 1))^{n_i} \end{aligned}$$

and

$$0 = \sum_i n_i dim_{\overline{\mathbf{Q}}_p} Ind_{G((E \cap F_\infty)/(N_i \cap F_\infty))}^{G((E \cap F_\infty)/F)}(\hat{\rho}_i) = \sum_i n_i [N_i \cap F_\infty : F]$$

so that

$$\prod_i \; H_{\rho_i}(u_{N_i}^s - 1)^{n_i} = (-1)^{\sum_i n_i} \in \{\pm 1\},$$

as required. □

9.4. Now let us examine the product

$$\prod_i \; h_{\rho_i}(u_{N_i}^s - 1)^{n_i}$$

of §9.2 when ρ is an arbitrary finite-dimensional $\overline{\mathbf{Q}}_p$-representation of $G(E/F)$. Recall from §8.1 that

$$h_{\rho_i}(T) = det(T + 1 - ((\gamma_{N_i}) \cdot -) \mid I_{E/N_i}(U(\rho_i)) \otimes_{\mathbf{Z}_p} \overline{\mathbf{Q}}_p)$$

where $U(\rho_i)$ is the underlying $G(E/N_i)$-modules of ρ_i. Hence

$$\begin{aligned} h_{\rho_i}(u_{N_i}^s - 1) &= det(u_{N_i}^s - ((\gamma_{N_i}) \cdot -) \mid I_{E/N_i}(U(\rho_i)) \otimes_{\mathbf{Z}_p} \overline{\mathbf{Q}}_p) \\ &= det(u_F^{s[N_i \cap F_\infty : F]} - ((\gamma_F)^{[N_i \cap F_\infty : F]} \cdot -) \mid I_{E/N_i}(U(\rho_i)) \otimes_{\mathbf{Z}_p} \overline{\mathbf{Q}}_p) \\ &= det(u_F^s - ((\gamma_F) \cdot -) \mid Ind_{G(F_\infty/N_i \cap F_\infty)}^{\Gamma}(I_{E/N_i}(U(\rho_i)) \otimes_{\mathbf{Z}_p} \overline{\mathbf{Q}}_p)), \end{aligned}$$

arguing as in the proof of Proposition 9.3.

Identifying $G(F_\infty/N_i \cap F_\infty)$ and $G(N_{i,\infty}/N_i)$, Proposition 8.5 implies that

$$h_{\rho_i}(u_{N_i}^s - 1) = h_{Ind_{G(E/N_i)}^{G(E/F)}(\rho_i)}(u_F^s - 1).$$

Therefore we have shown that

$$\begin{aligned} \prod_i \; h_{\rho_i}(u_{N_i}^s - 1)^{n_i} &= \prod_i \; h_{Ind_{G(E/N_i)}^{G(E/F)}(\rho_i)}(u_F^s - 1)^{n_i} \\ &= h_{\sum_i n_i Ind_{G(E/N_i)}^{G(E/F)}(\rho_i)}(u_F^s - 1) \\ &= h_\rho(u_F^s - 1). \end{aligned}$$

Let $\Omega_{\mathbf{Q}_p}$ denote the absolute Galois group, $G(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. Then $\Omega_{\mathbf{Q}_p}$ acts on $R(G(E/F))$, the ring of $\overline{\mathbf{Q}}_p$-representations of $G(E/F)$. If $\omega \in \Omega_{\mathbf{Q}_p}$ it is easy to see that $h_\rho(T) \in \mathcal{O}_K[T]$ satisfies $h_{\omega(\rho)}(T) = \omega(h_\rho(T))$ so that $h_\rho(T) \in \mathbb{Z}_p[\rho][T]$, where $\mathbb{Z}_p[\rho]$ is as in §9.1.

Also $L_p(1-n, \omega(\rho)) = \omega(L_p(1-n,\rho))$ for all integers $n \geq 2$. This is seen for one-dimensional ρ by writing the Artin L-function in the form (c.f. [71] (1.2) p.493)

$$L_F(1-n,\rho) = \sum_{\sigma \in G(E^{Ker(\rho)}/F)} \rho(\sigma)\zeta_F(\sigma, 1-n)$$

and observing that the partial zeta functions, $\zeta_F(\sigma, 1-n)$, are rational numbers. For an arbitrary ρ one uses Brauer's Induction Theorem and the inductivity properties of the L-function to reduce to the one-dimensional case (for a similar, but more complicated, calculation see [63] pp.28-30).

To recapitulate:

Theorem 9.5.

Let E/F be a Galois extension of totally real number fields with Galois group, $G(E/F)$, and let p be an odd prime. Let ρ be a finite-dimensional, irreducible $\overline{\mathbb{Q}}_p$-representation of $G(E/F)$. Let $h_\rho(T) \in \mathbf{Z}_p[\rho][T]$ be the characteristic polynomial defined in §8.1. Then there exists a unique unit power series

$$U_\rho(T) \in \mathbf{Z}_p[\rho][[T]]^*$$

such that the p-adic L-function of ρ satisfies, for all $s \in \mathbb{Z}_p$ (except $s = 0$ if ρ is trivial)

$$L_p(1-s,\rho) = \frac{\pi(\rho)^{\mu(\rho)} h_\rho(u^s - 1) U_\rho(u^s - 1)}{H_\rho(u^s - 1)},$$

where $H_\rho(T) = \rho(\gamma)(1+T) - 1$ if ρ is of type W and $H_\rho(T) = 1$ otherwise. Also $\pi(\rho) \in \mathbf{Z}_p[\rho]$ is a uniformiser and $u \in \mathbf{Z}_p^*$ is as in §9.1.

10. Determinantal congruences

10.1. Let p be an odd prime. If G is a finite group let $R(G)$ denote the complex representation ring of G (i.e. $R(G) = K_0(\mathbf{C}[G])$). Let $N/\mathbf{Q}_p$ be a Galois extension containing all the $|G|$-th roots of unity. If $\chi \in R(G)$ and $g \in G$ we denote by $\chi(g)$ the value of the character function of χ on g. We say that $\chi \equiv 0$ (modulo p^t) if $\chi(g) \in p^t\mathcal{O}_N$ for all $g \in G$.

The absolute Galois group of $\mathbf{Q}_p$, $\Omega_{\mathbb{Q}_p}$, acts on $R(G)$ in a manner which is characterised by the formula $\omega(\chi)(g) = \omega(\chi(g))$ for all $\omega \in \Omega_{\mathbf{Q}_p}$, $\chi \in R(G)$ and $g \in G$. This makes sense because $\chi(g) \in N$. Hence we may consider the group of $\Omega_{\mathbf{Q}_p}$-equivariant homomorphisms from $R(G)$ to $\mathcal{O}_N$, $Hom_{\Omega_{\mathbf{Q}_p}}(R(G), \mathcal{O}_N)$.

Define a sugroup $Cong_p(G) \subseteq Hom_{\Omega_{\mathbb{Q}_p}}(R(G), \mathcal{O}_N)$, by

$$Cong_p(G) = \{f \mid f(\chi) \in p^{t+1}\mathcal{O}_N \ if \ \chi \equiv 0 \ (modulo\ p^t)\}.$$

Define $\mathbb{Q}_p\{G\}$ to be the $\mathbb{Q}_p$-vector space on the conjugacy classes of elements of G. There is a canonical map of vector spaces, $c : \mathbb{Q}_p[G] \longrightarrow \mathbb{Q}_p G$, sending g to its conjugacy class. Write $p^e\mathbb{Z}_p\{G\}$ for $c(p^e\mathbb{Z}_p[G])$ for $e \geq 0$. There is an isomorphism ([62] §4.5.14 p.141; [63] §3.1.4 p.69)

$$\psi : \mathbb{Q}_p\{G\} \xrightarrow{\cong} Hom_{\Omega_{\mathbb{Q}_p}}(R(G), N)$$

given by $\psi(\sum_{g\in G} \lambda_g g)(\chi) = \sum_{g\in G} \lambda_g \chi(g)$ for $\lambda_g \in \mathbf{Q}_p$, $g \in G$ and $\chi \in R(G)$.

Consider the group, $Hom_{\Omega_{\mathbb{Q}_p}}(R(G), \mathcal{O}_N^*)$, of $\Omega_{\mathbb{Q}_p}$-equivariant homomorphism from $R(G)$ to the units in the integers of N. There is a homomorphism

$$Det : \mathbf{Z}_p[G]^* \longrightarrow Hom_{\Omega_{\mathbb{Q}_p}}(R(G), \mathcal{O}_N^*)$$

characterised by the formula

$$Det(\sum_{g\in G} \lambda_g g)(\rho) = det(\sum_{g\in G} \lambda_g \rho(g))$$

for all representations of the form $\rho : G \longrightarrow GL_m\mathcal{O}_N$ where $u = \sum_{g\in G} \lambda_g g$ is a unit in $\mathbb{Z}_p[G]$.

The image of Det is called the subgroup of determinantal homomorphisms.

Let $\psi^p : R(G) \longrightarrow R(G)$ denote the p-th Adams operations, characterised by $\psi(\chi)(g) = \chi(g^p)$. If $u \in \mathbb{Z}_p[G]^*$ then $Det(u)(\psi^p(\chi) - p\chi) \in 1 + p\mathcal{O}_N \subset \mathcal{O}_N^*$ for all $\chi \in R(G)$ ([62] §4.3.10 p.121; [63] §3.1.1 p.66). Hence $log_p(Det(u)(\psi^p(\chi) - p\chi)) \in p\mathcal{O}_N$ and the function, $\chi \mapsto log_p(Det(u)(\psi^p(\chi) - p\chi))$ lies in $Hom_{\Omega_{\mathbf{Q}_p}}(R(G), N) \cong \mathbb{Q}_p\{G\}$. The resulting homomorphism

$$L_G : \mathbf{Z}_p[G]^* \longrightarrow \mathbb{Q}_p\{G\}$$

is called the *group-ring logarithm*, originally constructed in a different manner by R. Oliver and M.J. Taylor, independently.

Now let us specialise to the case when G is a p-group. Then ([49]; [70]; see also [63] p.92) there is an exact sequence of the form

$$\mathbb{Z}_p[G]^* \xrightarrow{L_G} p\mathbb{Z}_p\{G\} \xrightarrow{\omega_G} G^{ab} \longrightarrow 0.$$

The homomorphism, ω_G, is due to R. Oliver and is described ([70] p.62). as the composition of $1/p$ times the abelianisation map with the projection

$$\mathbf{Z}_p[G^{ab}] \longrightarrow \frac{\mathbf{Z}_p + IG^{ab}}{\mathbf{Z}_p + (IG^{ab})^2} \cong G^{ab}.$$

Here $IG^{ab} \lhd \mathbf{Z}_p[G^{ab}]$ denotes the augmentation ideal and the final isomorphism sends the coset of $\sum_i a_i(g_i - 1)$ ($a_i \in \mathbf{Z}_p, g_i \in G^{ab}$) to the element, $\prod_i g_i^{a_i} \in G^{ab}$.

In general, by ([63] §§3.3.8-3.3.9) $\psi(pp\mathbb{Z}_p\{G\})$ is only a subgroup of $Cong_p(G)$ but when $G \cong \mathbb{Z}/p$ the $\psi(p\mathbf{Z}_p\{\mathbb{Z}/p\}) = Cong_p(\mathbb{Z}/p)$.

Theorem 10.2.

Let p be an odd prime and let G be a finite p-group of exponent p. Then there is an exact sequence of the form If l is an odd prime and G is a finite l-group then there is an exact sequence of the form

$$0 \longrightarrow Det(\mathbf{Z}_p[G]^*) \cap Hom_{\Omega_{\mathbf{Q}_p}}(R(G), \mu(p^\infty)) \longrightarrow Det(\mathbf{Z}_p[G]^*)$$

$$\stackrel{\psi L_G}{\longrightarrow} Cong_p(G) \stackrel{\omega_G \psi^{-1}}{\longrightarrow} G^{ab} \longrightarrow 0$$

where $\mu(p^\infty)$ denotes the group of p-primary roots of unity of N.

Proof

This is a consequence of the discussion of ([63] §3.3).

If $C \subseteq G$ is a cyclic subgroup, which is of order p by hypothesis, define $\Theta_C \in R(C)$ to be the unique element satisfying $\Theta_C(1) = 0$ and $\Theta_C(g) = p$ for non-trivial $g \in C$. If $f \in Hom_{\Omega_{\mathbb{Q}_p}}(R(G), \mathcal{O}_N)$ define $f_C \in Hom_{\Omega_{\mathbb{Q}_p}}(R(C), \mathcal{O}_N)$ by

$$f_C(\lambda) = [G : N_G C] f(Ind_C^G(\Theta_C \lambda)).$$

Hence $N_G C$ denotes the normaliser of C in G. If i_C denotes the inclusion of C into G then

$$f = \sum_{C \ cyclic} |G|^{-1} (i_C)_*(f_C) \in Hom_{\Omega_{\mathbb{Q}_p}}(R(G), \mathcal{O}_N)$$

where $(i_C)_*$ is the natural map induced by i_C. However, by ([63] §3.3.6 p.84), if $f \in Cong_p(G)$ then $f_C \in |G|^{-1} f_C \in Cong_p(C)$. Since each C has order p, by ([63] §3.3.8 p.86), $\psi(p\mathbb{Z}_p\{C\}) = Cong_p(C)$. By ([63] §3.3.1 p.81), $(i_C)_*$ maps $\psi(p\mathbb{Z}_p\{C\})$ to $\psi(p\mathbb{Z}_p\{G\})$ so that any $f \in Cong_p(G)$ lies in $\psi(p\mathbb{Z}_p\{G\})$.

Hence we have shown that ψ induces an isomorphism

$$\psi : p\mathbb{Z}_p\{G\} \stackrel{\cong}{\longrightarrow} Cong_p(G).$$

The result follows by substituting $Cong_p(G)$ into the exact sequence of ([63] §3.3.21 p.93). □

11. The conjecture

11.1. We return now to the situation of Theorem 9.5. Let $n \geq 2$ be an integer and let E/F be a Galois extension of totally real number fields with Galois group $G(E/F)$. In the notation of Theorem 9.5 and §10.1, the function

$$(\rho \mapsto U_\rho(u^{2n} - 1)) \in Hom_{\Omega_{\mathbb{Q}_p}}(R(G(E/F)), \mathcal{O}_N^*)$$

where, if $\rho = \sum_i n_i \rho_i \in R(G(E/F))$ and the ρ_i's are $\overline{\mathbb{Q}}_p$-representations,

$$U_\rho(u^{2n} - 1)) = \prod_i U_{\rho_i}(u^{2n} - 1))^{n_i}.$$

The remainder of this section will be devoted to a discussion of the following:

Conjecture 11.2. For each $n \geq 1$, there exists a unit $\alpha_{n,E/F} \in \mathbb{Z}_p[G(E/F)]^*$ such that

$$Det(\alpha_{n,E/F})(\rho) = U_\rho(u^{2n} - 1))$$

for all $\rho \in R(G(E/F))$.

11.3. *Evidence for Conjecture 11.2*

(i) **K-theory Galois module structure.**

Let G be a finite group and denote by $\mathcal{CL}(\mathbb{Z}[G])$ the class-group of finitely generated $\mathbb{Z}[G]$-modules of finite projective dimension so that $\mathcal{CL}(\mathbb{Z}[G]) = ker(rank : K_0(\mathbb{Z}[G]) \longrightarrow \mathbb{Z}$.

Suppose that

$$X \longrightarrow A \longrightarrow B \longrightarrow Y$$

is a 2-extension of finitely generated $\mathbb{Z}[G]$-modules representing an element of $Ext^2_{\mathbb{Z}[G]}(Y, X)$ inducing, via cup-product, isomorphisms between the Tate cohomology groups $H^i(G; Y)$ and $H^{i+2}(G; X)$ for all i. Under these circumstances a representative 2-extension may be chosen for which A and B are finitely generated $\mathbb{Z}[G]$-modules of finite projective dimension. The Euler characteristic of such a 2-extension is defined to be the element

$$[A] - [B] \in \mathcal{CL}(\mathbb{Z}[G]).$$

Here, for example, the class $[A]$ is defined to be the alternating sum of the finitely generated projectives in a finite resolution of A.

In the case when G is equal to $G(L/K)$, the Galois group of an extension of global fields, L/K, interesting examples of such Euler characteristics are afforded by the invariants, $\Omega(L/K, m)$ ($m = 1, 2, 3$), constructed in [22]. Further examples, denoted by $\Omega_1(L/K, 3)$, were constructed in ([63] Chapter 7) from 2-extensions in which Y and X are the algebraic K-groups of the ring of S-integers of L in dimensions 2 and 3, respectively. Here S is any finite $G(L/K)$-invariant set of finite primes containing those over primes of K which ramifiy in L/K, the construction being independent of the choice of S. In [23] this construction is generalised using K-groups in dimensions $2s$ and $2s + 1$ to yield a family of Euler characteristics, denoted by $\Omega_s(L/K, 3)$ for $s \geq 1$, to complete a pattern in which the original Chinburg invariant rightfully fits as $\Omega_0(L/K, 3)$.

On the other hand, associated to this situation, there is the Cassou-Noguès-Fröhlich class

$$W_{L/K} \in \mathcal{CL}(\mathbf{Z}[G(L/K)])$$

which is defined in terms of the root numbers of the L-functions of irreducible symplectic representations of $G(L/K)$ ([21]; [22]; [63]; [62]).

Surprisingly, in the case of function fields in characteristic p, one can show that $W_{L/K} = \Omega_s(L/K, 3)$ [23]. In the totally real number field case one show that a similar relation holds in the classgroup of a maximal order of the rational group-ring of $G(L/K)$ ([23]; [63] Chapter 7) and one expects this equation to hold in $\mathcal{CL}(\mathbb{Z}[G(L/K)])$ in general. In fact, this conjectured equation is borne out by the quaternionic calculations which we have managed to complete [25] [26].

Conjecture 11.2 arose out of the following approach to the proof that $W_{L/K} = \Omega_s(L/K, 3)$. There is an isomorphism, called the Fröhlich Hom-description of the class-group,

$$Det : \mathcal{C}L(\mathbb{Z}[G]) \xrightarrow{\cong} \frac{Hom_{\Omega_{\mathbb{Q}}}(R(G), J^*(K))}{Hom_{\Omega_{\mathbb{Q}}}(R(G), K^*) \cdot Det(U(\mathbb{Z}[G]))}$$

in which $J^*(K)$ denotes the group of idèles of a suitably Galois large Galois extension, $K/\mathbb{Q}$, and $Det(U(\mathbb{Z}[G]))$ denotes the unit idèles of the integral group-ring of G.

In particular, returning to the case when $G = G(E/F)$ as in Conjecture 5.2,

$$\Omega_n(E/F, 3) \in \mathcal{C}L(\mathbb{Z}[G(E/F)])$$

the element of the class-group may be represented an idèlic-valued function on $R(G(E/F))$. Using Iwasawa theory and the results of [5] one may imitate the proof in the function field case [24] to show that the p-adic coordinate, for p odd, of a Hom-description representative is given by the function $(\rho \mapsto h_\rho(u^{2n} - 1))$ of §11.1, providing that the appropriate Iwasawa μ-invariant vanishes (as is widely believed). If Conjecture 11.2 were true this would imply that another Hom-desription representative is given by $(\rho \mapsto L_p(1 - 2n, \rho)$ in the p-adic coordinate, for p odd, and this is sufficient to imply that $W_{E/F} = \Omega_n(E/F, 3) \in \mathcal{C}L(\mathbb{Z}[1/2][G(E/F)])$ for all $n \geq 1$.

(ii) **The abelian case of degree prime to p.**

When $G(E/F)$ is abelian Theorem 9.5 implies that $U_\rho(u^{2n} - 1))$ which implies that there exists a unit, $\alpha_{n,E/F}$, in the maximal $\mathbf{Z}_p$-order of $\mathbb{Q}_p[G(E/F)]$ such that

$$Det(\alpha_{n,E/F})(\rho) = U_\rho(u^{2n} - 1)$$

for all $\rho \in R(G(E/F))$. Further more, if p does not divide $[E : F]$ then this maximal order is equal to $\mathbb{Z}_p[G(E/F)]$, so that Conjecture 11.2 is true in this simple case.

(iii) **The case when $[E : F] = p$.**

It seems that, in this case, the results of [71] may be adapted to prove Conjecture 11.2 in lots of cases.

We close this section with an equivalent formulation of Conjecture 11.2 in a simple case.

Theorem 11.4.

Let p be an odd prime and let E/F be a Galois extension of totally real number fields whose Galois group $G(E/F)$ is a p-group of exponent p. Let $f(\rho) = log_p U_{(\psi^p(\rho)-p\rho)}(u^{2n}-1)$, when defined.

Then Conjecture 11.2 implies that

$$U_{(\psi^p(\rho)-p\rho)}(u^{2n}-1) \in 1 + p^{t+1}\mathcal{O}_N \subset \mathcal{O}_N^*$$

whenever $\rho \equiv 0$ (modulo p^t) and, when $[E:F]=p$, $f(\rho)$ is defined and satisfies

$$((p-1)/2)(f(1)/p) + Trace_{\mathbb{Q}_p(\xi_p)/\mathbb{Q}_p}((\xi_p/(1-\xi_p))(f(\chi)/p)) \in p\mathbb{Z}_p$$

for all faithful $\chi : G(E/F) \cong \mathbb{Z}/p \longrightarrow \mathbb{C}^*$.

Conversely, if the above two conditions hold then there exists a homomorphism, $h \in Hom_{\Omega_{\mathbb{Q}_p}}(R(G(E/F)), \mu(p^\infty))$ and a unit, $\alpha_{n,E/F} \in \mathbb{Z}_p[G(E/F)]^*$, such that

$$Det(\alpha_{n,E/F})(\rho)h(\rho) = U_\rho(u^{2n}-1)$$

for all $\rho \in R(G(E/F))$.

Proof

If $(\rho \mapsto U_{(\psi^p(\rho)-p\rho)}(u^{2n}-1))$ is equal to $Det(\alpha_{n,E/F})$ then it satisfies the determinantal congruences by ([62] §4.3.37; [63] §3.1.12 p.73). In this case composition with the p-adic logarithm gives a function

$$(\rho \mapsto f(\rho) = log_p(U_{(\psi^p(\rho)-p\rho)}(u^{2n}-1))) \in Cong_p(G(E/F)).$$

By Theorem 10.2, this function vanishes under $\omega_G\psi^{-1} : Cong_p(G(E/F)) \longrightarrow G(E/F)^{ab}$. However, by naturality of $U_\rho(T)$ this will happen if and only if the vanishing is true when E/F is replaced by the intermediate abelian extension, M/F corresponding to the abelian quotient $G(E/F)^{ab} \cong G(M/F)$. Since $G(E/F)$ has exponent p, $G(M/F)$ is an elementary abelian p-group. Therefore $G(M/F)$ is faithfully detected by all the quotient maps, $G(M/F) \longrightarrow \mathbf{Z}/p$. Hence, by naturality again, $\omega_G\psi^{-1}(f)$ vanishes if and only if it does so in the case when $G(E/F) \cong \mathbf{Z}/p$.

Now we shall calculate $w_{\mathbf{Z}/p}\psi^{-1}(f)$ for $f \in Cong_p(\mathbf{Z}/p)$, assuming that $[E:F]=p$. To accomplish this we must recall the formula for $\phi = \psi^{-1}$ from ([63] p.80). Let g be a generator of the cyclic group of order p and let χ be the faithful one-dimensional representation given by $\chi(g) = exp(2\pi i/p) = \xi_p$. Then the elements of $R(\mathbf{Z}/p) \otimes \mathbf{C}$ which are dual to the powers of g are

$$\hat{\gamma}_s = p^{-1}\sum_{i=0}^{p-1} \chi(g^{-is})\chi^i$$

for $0 \leq s \leq p-1$, since

$$\hat{\gamma}_s(g^j) = p^{-1}\sum_{i=0}^{p-1} \chi(g^{i(j-s)}) = \begin{cases} 1 & if\ j = s, \\ 0 & otherwise. \end{cases}$$

Then, by ([63] p.80),

$$\phi(f) = \sum_{s=0}^{p-1}\sum_{i=0}^{p-1} p^{-1}\chi(g^{-is})f(\chi^i)g^s \in \mathbf{Z}_p[\mathbf{Z}/p].$$

Hence

$$\omega_{\mathbf{Z}/p}(\phi(f)) \quad = \prod_{s=0}^{p-1}(g^s)^{(1/p)\sum_{i=0}^{p-1} p^{-1}\chi(g^{-is})f(\chi^i)} = g^{(1/p^2)\sum_{s=0}^{p-1}\sum_{i=0}^{p-1} s\chi(g^{-is})f(\chi^i)}.$$

However

$$\begin{aligned}\sum_{s=0}^{p-1}\sum_{i=0}^{p-1} s\chi(g^{-is})f(\chi^i) &= \sum_{s=1}^{p-1} sf(1) + \sum_{i=1}^{p-1}\sum_{s=1}^{p-1} s\xi_p^{-is}f(\chi^i) \\ &= (p(p-1)/2)f(1) + Trace_{\mathbf{Q}_p(\xi_p)/\mathbf{Q}_p}((\sum_{s=1}^{p-1} s\xi_p^{-s})f(\chi))\end{aligned}$$

and

$$\begin{aligned}\sum_{s=1}^{p-1} s\xi_p^{-s} &= (x\tfrac{d}{dx}(1+x+\ldots+x^{p-1}))|_{x=\xi_p^{-1}} \\ &= \xi_p^{-1}\prod_{j=2}^{p-1}(\xi_p^{-1}-\xi_p^{-j}) \\ &= \xi_p^{-(p-1)}\prod_{j=2}^{p-1}(1-\xi_p^{-j+1}) \\ &= \xi_p p/(1-\xi_p).\end{aligned}$$

Therefore we have

$$\omega_{\mathbf{Z}/p}(\phi(f)) = g^{((p-1)/2)(f(1)/p)+Trace_{\mathbf{Q}_p(\xi_p)/\mathbf{Q}_p}((\xi_p/(1-\xi_p))(f(\chi)/p))}.$$

This completes the proof in one direction. Conversely the discussion shows that if the two conditions hold then there exists a unit, $\alpha_{n,E/F} \in \mathbf{Z}_p[G(E/F)]^*$, such that

$$log_p Det(\alpha_{n,E/F})(\psi^p\rho - p\rho) = log_p U_{\psi^p\rho - p\rho}(u^{2n}-1) \in p\mathcal{O}_N.$$

Therefore $Det(\alpha_{n,E/F})(\psi^p\rho - p\rho)/U_{\psi^p\rho-p\rho}(u^{2n}-1))$ is a p-primary root of unity which is congruent to 1 modulo p (cf. [?] §4.5.31). Since $G(E/F)$ has exponent p, $\psi^p(\rho) = dim(\rho)$ and therefore $Det(\alpha_{n,E/F})(\rho)/U_\rho(u^{2n}-1))^p$ is equal to $Det(\alpha_{n,E/F})(dim(\rho))/U_{dim(\rho)}(u^{2n}-1))$ times a p-primary root of unity which is congruent to 1 modulo p. On the other hand, setting $\rho = 1$ we set that $Det(\alpha_{n,E/F})(1)/U_1(u^{2n}-1))^{p-1}$ is also a p-primary root of unity which is congruent to 1 modulo p, since $\psi^p(1) - p = 1-p$. Since $Det(\alpha_{n,E/F})(1)/U_1(u^{2n}-1))$ is congruent to 1 modulo p then it must also be a p-primary root of unity, which completes the proof. □

References

[1] N. Abe, G. Henniart, F. Herzig and M-F. Vigneras: A classification of irreducible admissible mod p representations of p-adic reductive groups; J.A.M.Soc. 30 #2 (2017) 495-559. (http://dx.doi.org/ 10.1090/jams/862.)

[2] Eran Assaf, David Kazhdan and Ehud de Shalit: Kirillov models and the Breuil-Schneider conjecture for $GL_2(F)$; arXiv:1302.3060.2013.

[3] B. Aupetit and H. du T. Mouton: Trace and determinant in Banach algebras; Studia Mathematica 121 (2) 114-136 (1996).
[4] D. Barsky: Fonctions Zeta p-adiques d'une Classe de Rayon des Corps de Nombres Totalement-réels; Groupe d'étude d'analyse ultrametrique (1977-78).
[5] P. Bayer and J. Neukirch: On values of zeta functions and l-adic Euler characteristics; Inventiones Math. 50 (1978) 35-64.
[6] J. Bernstein and A. Zelevinski: Representations of the group $GL(n, F)$ where F is a local non-Archimedean field; Uspekhi Mat. Nauk. **31** 3 (1976) 5-70.
[7] J. Bernstein and A. Zelevinski: Induced representations of reductive p-adic groups I; Ann. ENS **10** (1977) 441-472.
[8] R. Boltje: A canonical Brauer induction formula; Astérisque 181-182 (1990) 31-59.
[9] R. Boltje: Monomial resolutions; J.Alg. 246 (2001) 811-848.
[10] N. Bourbaki: *Algèbre*; Hermann (1958).
[11] N. Bourbaki: *Mésures de Haar*; Hermann (1963).
[12] N. Bourbaki: *Variétés différentielles et analytiques*; Hermann (1967).
[13] N. Bourbaki: *Groupes et algèbres de Lie*; Hermann (1968).
[14] N. Bourbaki: Groupes er algèbres de Lie; Chapters IV-VI (1968) Hermann Paris.
[15] F. Bruhat: Sur les représentations induites des groupes de Lie; Bull. Soc. Math. France 84 (1956) 97-205.
[16] F. Bruhat: Distributions sur un groupe localement compact et applications à l'étude des représentations des groupes p-adiques; ; Bull. Soc. Math. France 89 (1961) 43-75.
[17] F. Bruhat and J. Tits: Groupes algébriques simples sur on corps local; Driebergen Conference on Local Fields, Springer-Verlag (1967).
[18] C.J. Bushnell and G. Henniart: *The Local Langlands Conjecture for $GL(2)$*; Grund. Math. Wiss. #335; Springer Verlag (2006).
[19] Daniel Bump: *Automorphic forms and representations*; Cambridge studies in advanced math. **55** (1998).
[20] P. Cassou-Nogués: Valeurs aux entiéres negatif des fonctions zeta et fonctions zeta p-adique; Inventiones Math. 51 (1979) 29-59.
[21] Ph. Cassou-Nogués, T. Chinburg, A. Fröhlich and M.J. Taylor: L-functions and Galois modules; London Math. Soc. 1989 Durham Symposium,*L-functions and Arithmetic* Cambridge University Press (1991) 75-139.
[22] T. Chinburg: Exact sequences and Galois module structure; Annals Math. 121 (1985) 351-376.
[23] T. Chinburg, M. Kolster, G. Pappas and V.P. Snaith: Galois module structure of K-groups of rings of integers; C.R. Acad. Sci. Paris t.320 Series I (1995) 1435-1440.
[24] T. Chinburg, M. Kolster, G. Pappas and V.P. Snaith: Galois module structure of K-groups of rings of integers; to appear in K-theory.
[25] T. Chinburg, M. Kolster, G. Pappas and V.P. Snaith: Quaternionic exercises in K-theory Galois module structure; Proc. Great Lakes K-theory Conf., Fields Institute Communications Series #16 (A.M.Soc. Publications) 1-29 (1997).
[26] T. Chinburg, M. Kolster and V.P. Snaith: Quaternionic exercises in K-theory Galois module structure II; to appear in Proc. Algebraic K-theory Conf. I.C.T.P. Trieste (1997) Birkhäuser.
[27] J.H. Conway, R.T. Curtis, S.P. Norton and R.A. Wilson: *Atlas of Finite Groups*; Clarendon Press, Oxford (1985).
[28] Ana Cariani, Matthew Emerton, Toby Gee, David Geraghty, Vyautas Paskunas and Sug Woo Shin: Patching and the p-adic Langlands correspondence; Cambridge J. Math (2014).

[29] H. Davenport and H. Hasse: Die Nullstellen der Kongruenzzetafunktionen in gewissen zyklischen Fllen; Journal fr die Reine und Angewandte Mathematik **172** (1935) 151182.
[30] P. Deligne: Le "centre" de Bernstein; *Représentations des groupes réductifs sur un corps local* Travaux en cours, Hermann, Paris (1984) 1-32.
[31] P. Deligne and K. Ribet: Values of abelian L-functions at negative integers over totally real fields; Invent. Math. 59 (1980) 227-286.
[32] F. Gantmacher: Canonical representation of automorphisms of a complex semi-simple Lie group; Mat. Sb. vol.47 (1939).
[33] J.A. Green: The characters of the finite general linear groups; Trans. Amer. Math. Soc. 80 (1955) 402-447.
[34] R. Greenberg: On p-adic Artin L-functions; Nagoya J. Math. 89 (1983) 77-87.
[35] Florian Herzig: The classification of irreducible admissible mod p representations of a p-adic GL_n; Inv. Math. **186** (2011) 373-434.
[36] P.J. Hilton and U. Stammbach: *A Course in Homological Algebra*; GTM #4 (1971) Springer Verlag.
[37] M. I. Isaacs: *Characters of finite groups*; Dover (1994) ISBN 978-0-486-68014-9.
[38] N. Jacobson: *Lie Algebras*; Interscience (1962).
[39] G.D. James: The irreducible representations of the symmetric groups; Bull. London Math. Soc. 8 (1976) 229-232.
[40] G. D. James: The Representation Theory of the Symmetric Groups; Springer Verlag Lecture Notes in Math. #682.
[41] J. P. Keating, F. Mezzadri and B. Singphu: Rate of convergence of linear functions on the unitary group; J. Phys. A. Math. Theor. 44 (2011) o35204.
[42] T. Kondo: On Gaussian sums attached to the general linear groups over finite fields; J. Math. Soc. Japan vol.15 #3 (1963) 244-255.
[43] R.P. Langlands: Problems in the theory of automorphic forms; *Lectures in modern analysis and its applications III*, Springer LNMath #170 (1970) 18-61.
[44] I.G. Macdonald: Zeta functions attached to finite general linear groups; Math. Annalen 249 (1980) 1-15.
[45] J.P. May, V.P. Snaith and P. Zelewski: A further generalisation of the Segal conjecture; Quart. J. Math. Oxford (2) 40 (1989) 457-473.
[46] J.W. Milnor and J.C. Moore: On the structure of Hopf algebras; Annals of Math. (2) 81 (1965) 211-264.
[47] D. Montgomery and L. Zippin: *Topological Transformation Groups*; Interscience New York (1955).
[48] Olaf Neisse and Victor Snaith: Explicit Brauer Induction for symplectic and orthogonal representations; Homology, Homotopy Theory and Applications 7 (3) 2005, Conference edition "Applications of K-theory and Cohomology" (ed. J.F. Jardine).
[49] R. Oliver: SK_1 for finite group rings II; Math. Scand. 47 (1980) 195-231.
[50] J.M. O'Sullivan and C.N. Harrison: Myelofibrosis: Clinicopathologic Features, Prognosis and Management; Clinical Advances in Haematology and Oncology **16** (2) February 2018.1.
[51] C. Queen: The existence of p-adic L-functions; *Number Theory and Algebra*, Collected papers dedicated to Henry B. Mann (Academic Press, New York and London) 263-288 (1977).
[52] K. Ribet: Report on p-adic L-functions over totally real fields; Astérisque 61 (1979) 177-192.
[53] A. M. Robert: *A Course in p-Adic Analysis*; Grad Texts in Math. #198, Springer Verlage (2000).

[54] I. Satake: Theory of spherical functions on reductive algebraic groups over $\mathcal{P}$-adic fields; Pub. Math. no. 18 I.H.E.S. (1963).

[55] J-P. Serre: *Linear Representations of Finite Groups* ; Grad. Texts in Math. # 42 (1977) Springer-Verlag.

[56] J-P. Serre: Complète Réductibilité; Séminaire BOURBAKI 56ième année 2003-2004 #932 p. 195 à 217.

[57] T. Shintani: Two remarks on irreducible characters of finite general linear groups; J. Math. Soc. Japan **28** (1976) 396-414.

[58] Allan J. Silberger: The Langlands quotient theorem for p-adic groups; Math. Annalen 236 no. 2 (1978) 95-104.

[59] Allan J. Silberger: *Introduction to harmonic analysis on p-adic reductive groups*; Math. Notes Princeton Unviersity Press (1979).

[60] V.P. Snaith: Explicit Brauer Induction; Inventiones Math. 94 (1988) 455-478.

[61] V.P. Snaith: *Topological Methods in Galois Representation Theory*, C.M.Soc Monographs, Wiley (1989) (republished by Dover in 2013).

[62] V.P. Snaith: *Explicit Brauer Induction (with applications to algebra and number theory)*; Cambridge studies in advanced mathematics #40,Cambridge University Press (1994).

[63] V.P. Snaith: *Galois Module Structure*; Fields Institute Monographs # 2 A.M.Soc. (1994).

[64] V.P. Snaith: *Derived Langlands*; World Scientific (2018).

[65] V.P. Snaith: Derived Langlands II: HyperHecke algebras, monocentric relatons and $\mathcal{M}_{cmc,\underline{\phi}}(G)$-admissibility; arXiv:3100675 [math.RT] 24 Mar 2020.

[66] V.P. Snaith: Derived Langlands III: PSH algebras and their numerical invariants; arXiv:3223311 [math.RT] 12 June 2020.

[67] Derived Langlands IV: Notes on $\mathcal{M}_c(G)$-induced representations; arXiv:2008.06325v1 [math.RT] 14 Aug 2020.

[68] V.P. Snaith: Derived Langlands V: The Hopflike properties of the hyperHecke algebra; preprint on University of Sheffield homepage (27 May 2020).

[69] P. Symonds: A splitting principle for group representations; Comm. Math. Helv. 66 (1991) 169-184.

[70] M.J. Taylor: *Class Groups of Group Rings*; L.M.Soc. Lecture Notes Series # 91 (1984) Cambridge University Press.

[71] A. Wiles: The Iwasawa conjecture for totally real fields; Annals of Math. 131 (1990) 493-540.

[72] A.V. Zelevinsky: *Representations of finite classical groups - a Hopf algebra approach*; Lecture Notes in Math. #869, Springer-Verlag (1981).