GROUP THEORY AND THE 2×2 RUBIK'S CUBE

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ABSTRACT. This essay was motivated by my grandson Giulio being given one of these toys as a present. If I have not made errors the moves described here, particularly in §4, will suffice to solve the puzzle if all else fails!

1. SYMMETRIES OF THE SOLID CUBE

1.1. Consider the following three matrices

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \ Y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \ Z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrix products of each of these with itself satisfy

$$X^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \ Y^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \ Z^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence each of X, Y, Z has order four. Also we have identities

$$X^3 Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

and

$$ZY = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

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so that $ZYZ^{-1} = X^3$. Similarly we have

$$Z^3Y = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

and

$$YX = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

so that $YXY^{-1} = Z^3$. Finally we have

$$Y^3X = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$XZ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

so that $XZX^{-1} = Y^3$.

Now consider $ZXZ^{-1}, YZY^{-1}, XYX^{-1}$.

$$ZX = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$YZ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

so that $ZXZ^{-1} = Y$. Next

$$YZ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$XY = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

so that $YZY^{-1} = X$. Finally

$$XY = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$ZX = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

so that $XYX^{-1} = Z$.

1.2. *Recapitulation of the conjugation relations*

$$ZYZ^{-1} = X^3, \ YXY^{-1} = Z^3, \ XZX^{-1} = Y^3,$$

$$ZXZ^{-1} = Y, \ YZY^{-1} = X, \ XYX^{-1} = Z,$$

$$X^2Z^2 = Z^2X^2 = Y^2.$$

1.3. *The group generated by X, Y, Z*

The 3×3 permutation matrices are those 3×3 matrices which have only one non-zero entry in each row and column and this entry is a 1. There are $3! = 6$ of them. The $3\times$ matrices with only one non-zero entry which is ± 1 is the semi-direct product of the diagonal matrices whose entries are ± 1 with the permutation group on 3 objects, denoted by Σ_3. This semi-direct product has $3! \times 2 \times 2 \times 2 = 48$ elements whose determinants are ± 1. There are 24 matrices in this semi-direct product having determinant 1.

Now consider the group generated by X, Y, Z. There is a homomorphism from this group onto Σ_3 given by turning each non-zero entry of the matrix into a 1, which gives a permutation matrix. Therefore, as permutations of

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the set $\{1, 2, 3\}$ we have sent X to the permutation which switches 2 and 3, Y switches 1 and 3 and Z switches 1 and 2. These transpositions generate Σ_3 and the kernel of the surjective homomorphism $\langle X, Y, Z \rangle \longrightarrow \Sigma_3$ consists of the four diagonal matrices $1, X^2, Y^2 = X^2 Z^2, Z^2$. Therefore $\langle X, Y, Z \rangle$ has at least 24 elements and so it is equal to the group consisting of the 24 matrices in this semi-direct product having determinant 1.

1.4. *Pictures of the cube*

   Below is a picture of the cube with vertices labelled $A, B, C, D, \alpha, \beta, \gamma, \delta$. A symmetry of the cube is a way of moving the rigid cube so as to occupy exactly the same volume in space but perhaps with the vertices in different places. We shall now examine how to each assign to each symmetry of the cube a unique matrix among the 24 matrices in the group generated by X, Y, Z of §1.1 and §1.2.
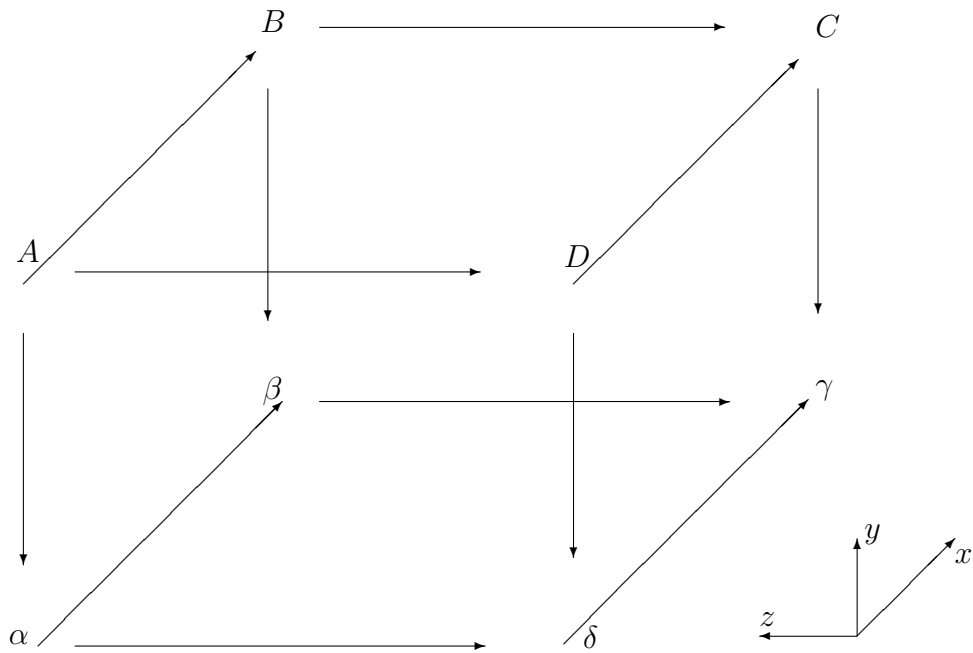
   The cube possesses three axes of symmetry. These are the lines which pass through the centres of a pair of opposite faces. We shall assign to each of these axes a direction along the axis. These shall be denoted by letters x, y, z. The direction from the midpoint of the face A, D, δ, α towards the midpoint of B, C, γ, β will be denoted by x. Similarly from the midpoint of the face D, C, γ, δ towards the midpoint of A, B, β, α will be denoted by z and from the midpoint of the face $\alpha, \beta, \gamma, \delta$ towards the midpoint of A, B, C, D will be denoted by y.

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Each symmetry of the cube sends the set of three axes into some permutation of themselves but not necessarily preserving the direction. Consider the following array

	x	y	z
x			
y			
z			

Given a symmetry of the cube. We shall record in the x-row what it does to the x-direction, in the y-row what it does to the y-direction and in the z-row what it does to the z-direction. Consider where it sends x. In the x-row put a ± 1 in the place in the column corresponding to the axis that x is sent to - putting a 1 if the direction is correct and put -1 there if the direction is the opposite. Do the same for the y and z columns and put zeroes in the other six entries in the array.

For example, let X be the rotation through $\pi/2$ radians in a clockwise direction looking along the direction x. This sends x to itself with the same direction, z to y with the same direction and y to z with the reversed direction. This gives an array which is the matrix X of §1.1.

X :

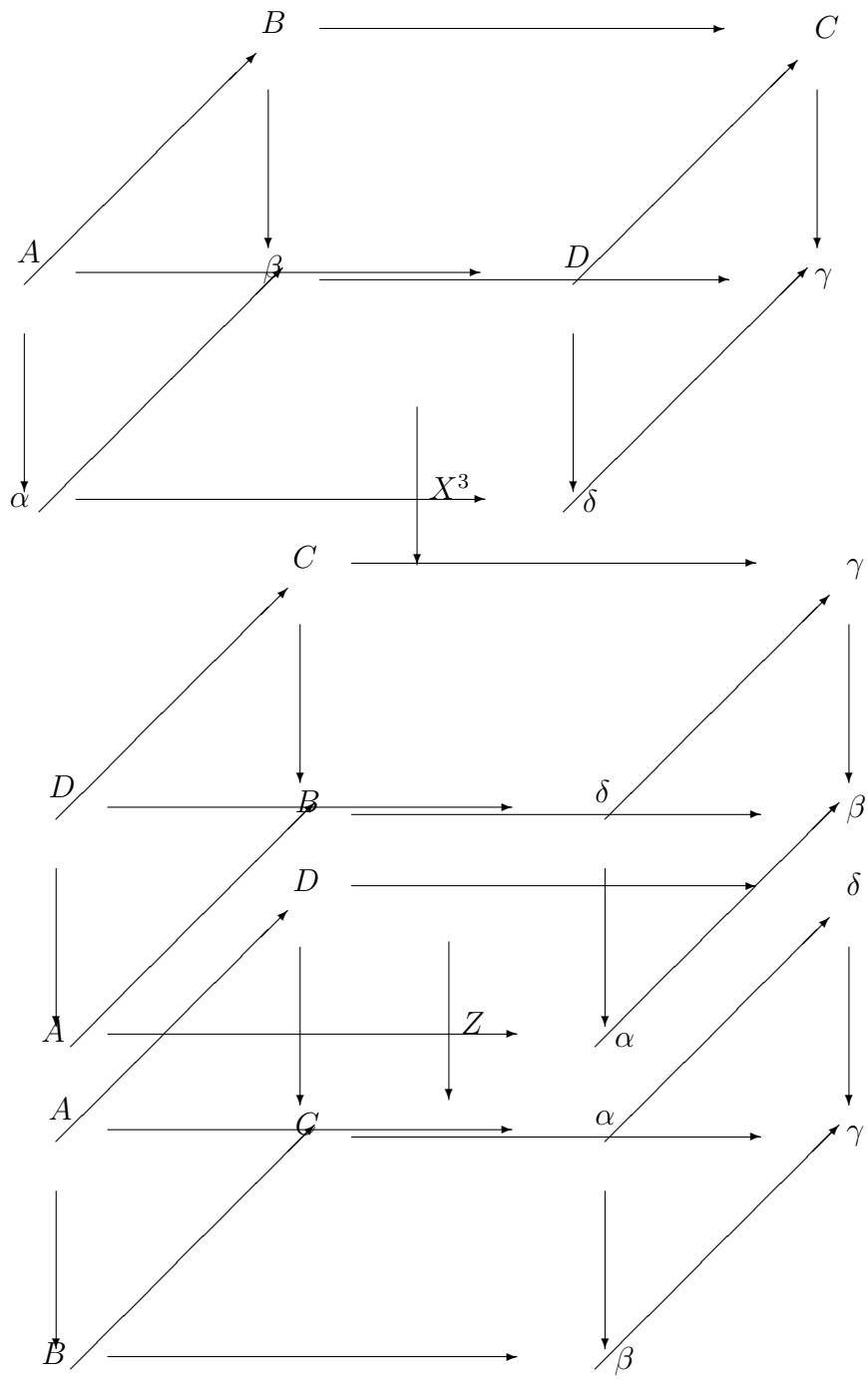|	x	y	z
x	1	0	0
y	0	0	-1
z	0	1	0

Similarly, if Y and Z are the rotations through $\pi/2$ radians in a clockwise direction looking along the direction y and z, respectively, we obtain arrays which coincide with the matrices Y and Z, respectively, of §1.1.

Y :

	x	z	y
x	0	0	1
z	0	1	0
y	-1	0	0

Z :

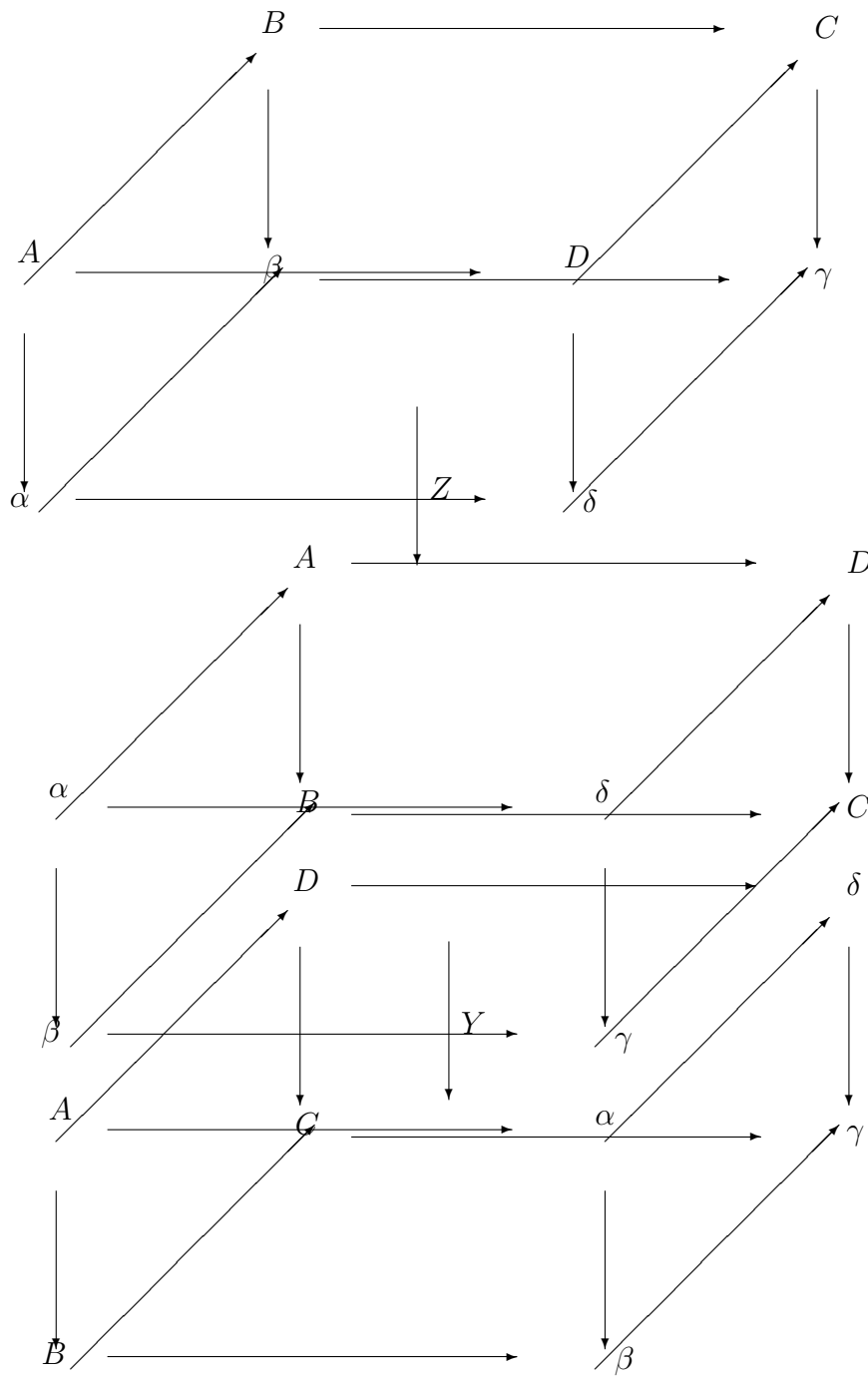|	x	z	y
x	0	-1	0
z	1	0	0
y	0	0	1

If U and V are two symmetries of the cube their product UV means the symmetry given by first performing U and then performing V to the result of U. For example, $X^3Y = XXXY$ means three performances of X followed by one of Y. This product, when we interpret symmetries as 3×3 matrices, corresponds exactly with the product of matrices. In general sending the symmetries X, Y, Z to the matrices X, Y, Z yields an isomorphism between the group of symmetries of the solid cube with the product described above and the group of matrices generated by X, Y, Z with product given by matrix multiplication.

For example, consider the matrix product relation $X^3Z = ZY$ of §1.2. Since X^3 is the opposite rotation to X we may depict X^3Z in the following diagrammatic form:

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X^3Z

ZY

$B \longrightarrow C$

A $\beta \longrightarrow D$ γ

$\alpha \longrightarrow Z \longrightarrow \delta$

$A \longrightarrow D$

α $B \longrightarrow \delta$ C

$D \longrightarrow \delta$

$\beta \longrightarrow Y \longrightarrow \gamma$

A $C \longrightarrow \alpha$ γ

$B \longrightarrow \beta$

2. 2×2 RUBIK SYMMETRIES

2.1. We have seen how the symmetries of a solid cube permute the eight vertices $A, B, C, D, \alpha, \beta, \gamma, \delta$. The basic Rubik moves twist half the cube through $\pi/2$ and which also permutes the vertices.

First chose a directed axis e.g. x. Sight along the chosen axis x in the x-direction and then rotate the half of the cube further in the x-direction clockwise through $\pi/2$. This Rubik move will be denoted by X_+ and the same rotation applied to the other half of the cube will be denoted by X_-. Similarly we can define Y_+, Y_-, Z_+, Z_-. These moves satisfy

$$X = X_+ X_- = X_- X_+, X_+^4 = X_-^4 = 1,$$

$$Y = Y_+ Y_- = Y_- Y_+, Y_+^4 = Y_-^4 = 1,$$

$$Z = Z_+ Z_- = Z_- Z_+, Z_+^4 = Z_-^4 = 1.$$

From $ZYZ^{-1} = X^{-1}$ we find that

$$Z_- Y Z_-^{-1} = (Z_+ X Z_+^{-1})^{-1}, \ Z_+ Y Z_+^{-1} = (Z_- X Z_-^{-1})^{-1}$$

and $XYX^{-1} = Z$ implies

$$X_- Y X_-^{-1} = X_+ Z X_+^{-1}, \ X_+ Y X_+^{-1} = X_- Z X_-^{-1}.$$

3. CONJUGATION BY X, Y, Z

3.1. In this section we calculate some conjugates starting with $X^{-1} g X$ for some g's. Recall that, as a 3×3 matrix
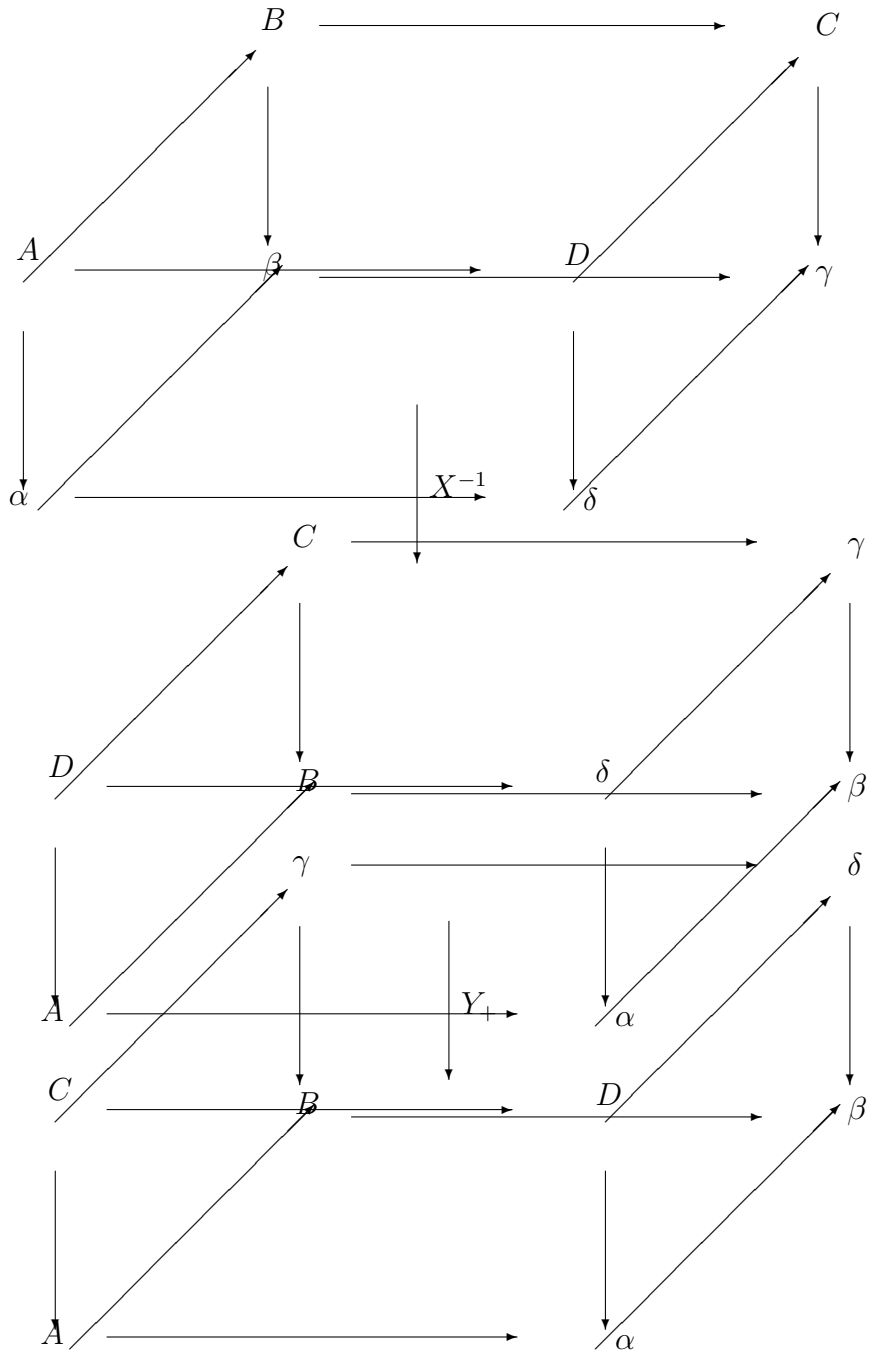
$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

According to the recipe of §1.4, this means that X preserves the x-axis and its direction, sends the y-axis to the reversed z-axis and sends the z-axis to the y-axis preserving positive directions.
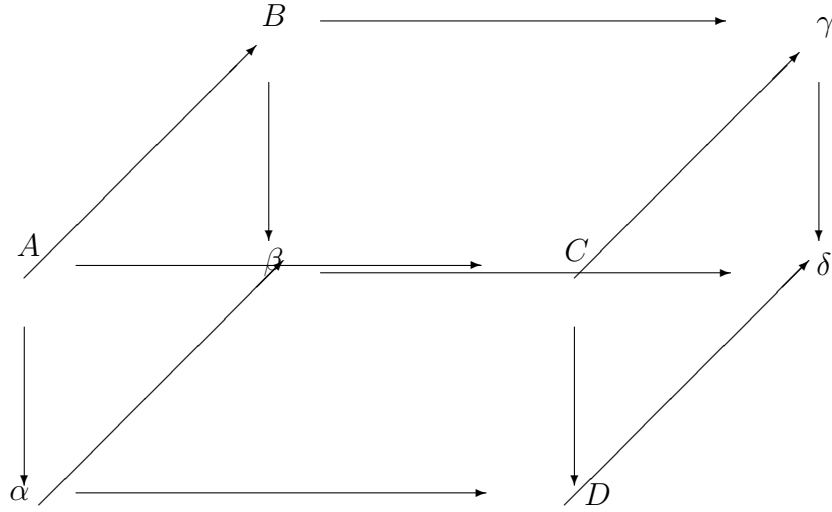
Therefore $X^{-1} Y_+ X$ may be depicted by the following diagrams:

$X^{-1}Y_+$

Therefore the final result of $X^{-1}Y_+X$ is:

Hence we have $X^{-1}Y_+X = Z_-^{-1}$. Since $Y = Y_-Y_+$ and $XYX^{-1} = Z$ so that $X^{-1}YX = (X^{-1}Y_+X)(X^{-1}Y_-X) = Z_-^{-1}X^{-1}Y_-X$. On the other hand $X^2ZX^2 = X^{-1}YX$ and

$$X^2ZX^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = Z^{-1}.$$

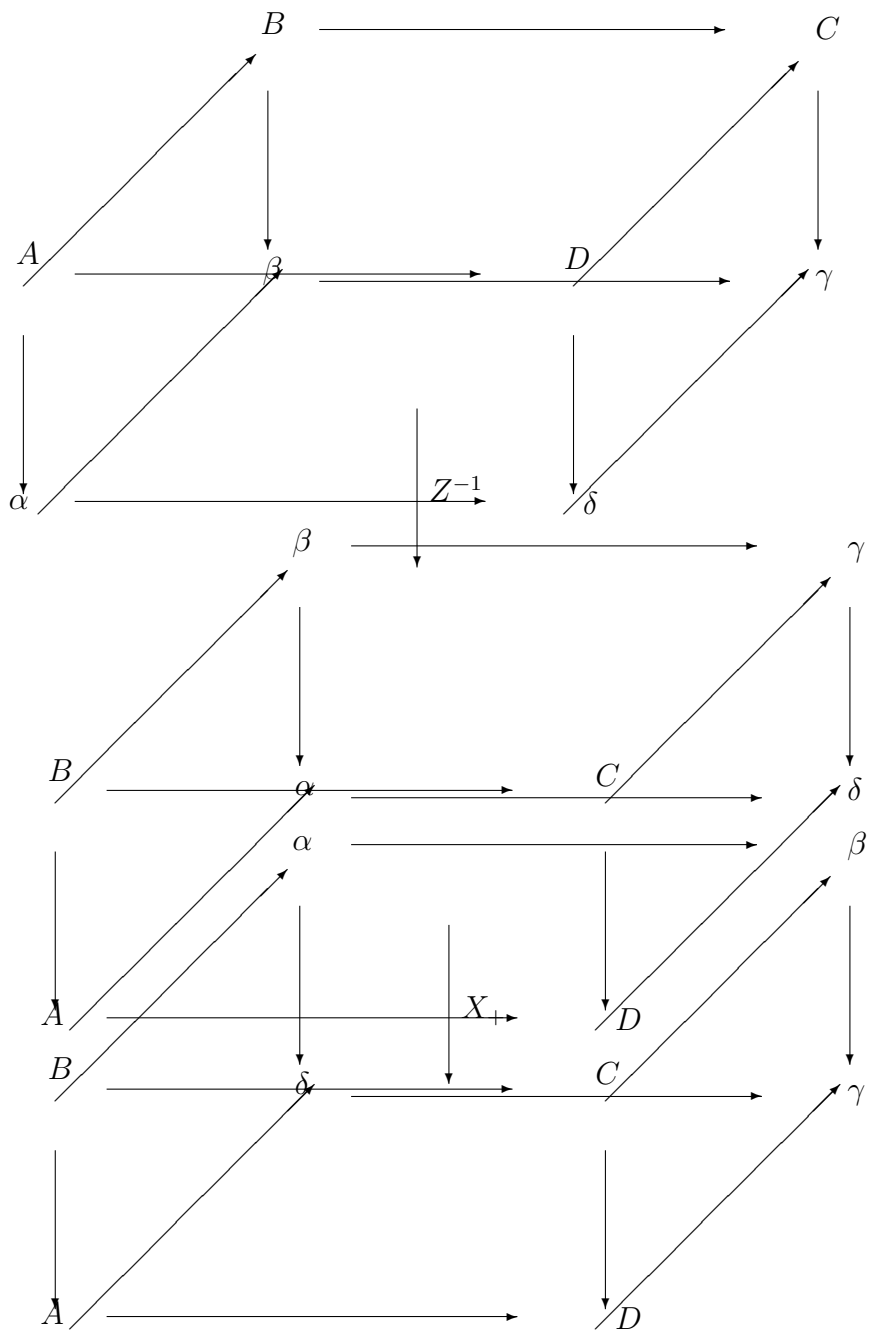Therefore $X^{-1}Y_-X = Z_+^{-1}$. In addition $X^{-1}X_+X = X_+$ and $X^{-1}X_-X = X_-$.

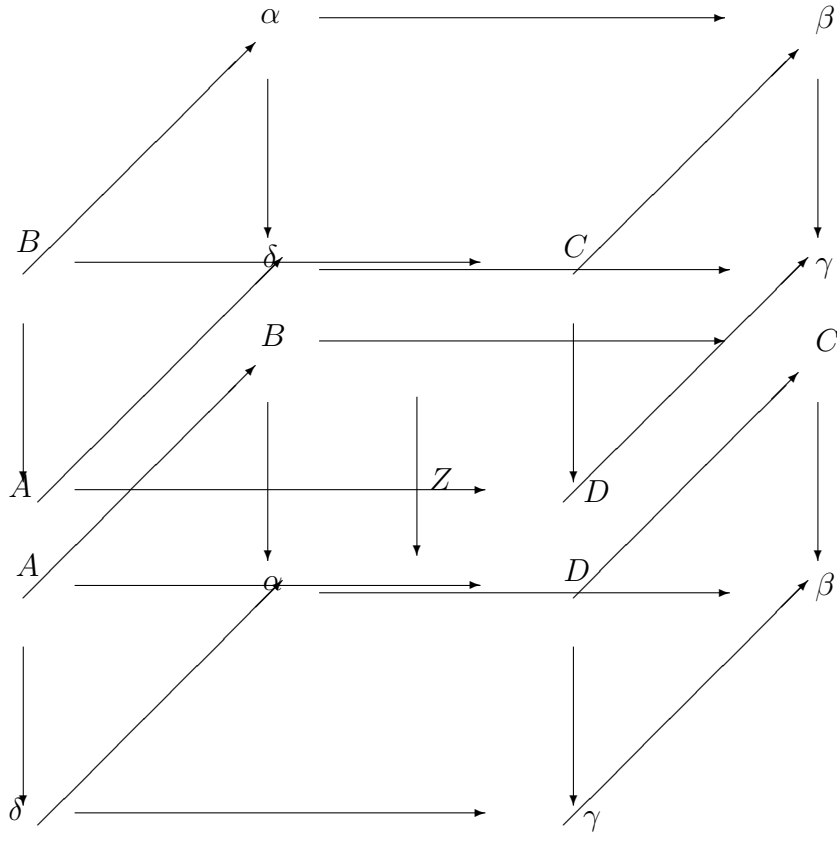Next we shall compute $Z^{-1}gZ$ for some g's. Recall that, as a 3×3 matrix

$$Z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

According to the recipe of §1.4, this means that Z preserves the z-axis and its direction, sends the x-axis to the reversed y-axis and sends the y-axis to the x-axis preserving positive directions.

Therefore $Z^{-1}X_+Z$ may be depicted by the following diagrams:

$Z^{-1}X_+$

$B \longrightarrow C$

$A \qquad \beta \longrightarrow D \qquad \gamma$

$\alpha \qquad Z^{-1} \qquad \delta$

$\beta \qquad \gamma$

$B \qquad \alpha \longrightarrow C \qquad \delta$

$\alpha \longrightarrow \beta$

$A \qquad X_+ \qquad D$

$B \qquad \delta \longrightarrow C \qquad \gamma$

$A \longrightarrow D$

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Therefore we have $Z^{-1}X_+Z = Y_-^{-1}$. This implies that

$$Z^2 Y Z^2 = Z^2 Z X Z^{-1} Z^2 = Z^{-1} X Z = Z^{-1} X_- Z Y_-^{-1}$$

However we have

$$Z^2 Y Z^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = Y^{-1}.$$

Therefore $Z^{-1}X_-Z = Y_+^{-1}$. Also $Z^{-1}Z_-Z = Z_-$ and $Z^{-1}Z_+Z = Z_+$.

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4. The commutators

4.1. $Z_-^{-1}X_-^{-1}Z_-X_-$

Recall that a composition of symmetries UV means, in this essay, first perform U and then perform V. Therefore the commutator $Z_-^{-1}X_-^{-1}Z_-X_-$ means first perform the inverse of Z_-, then the inverse of X_-, then Z_- and finally X_-.

Just to recapitulate X_- rotates the front half of the cube by $\pi/2$ clockwise in the x direction. Also Z_- is rotation of the right half of the cube by $\pi/2$ clockwise in the z direction.

Below I am going to depict by a sequence of four pictures the effect of $Z_-^{-1}X_-^{-1}Z_-X_-$ on each of the eight corner sub-cubes.
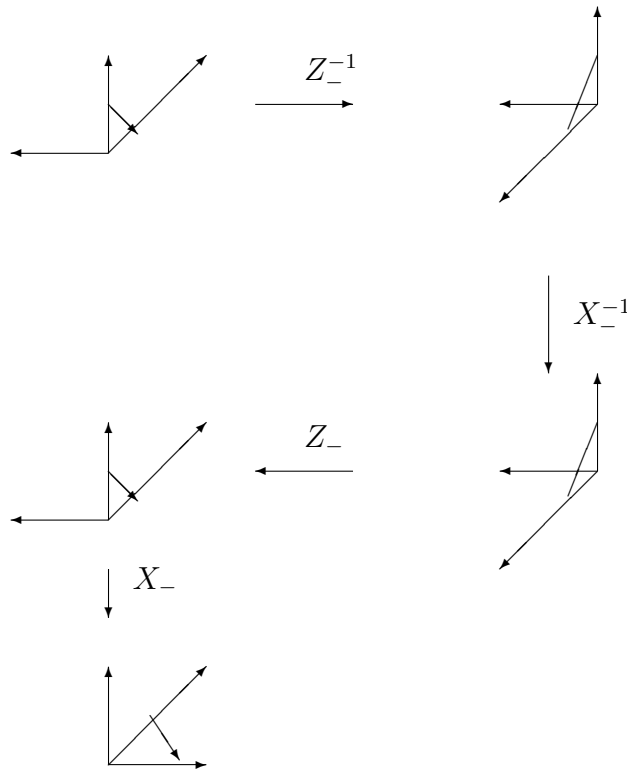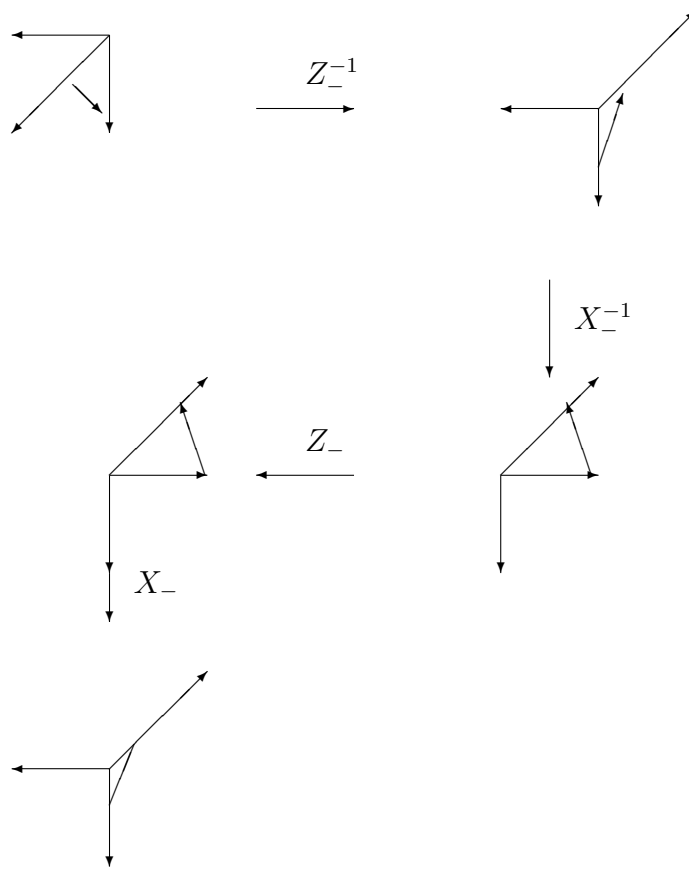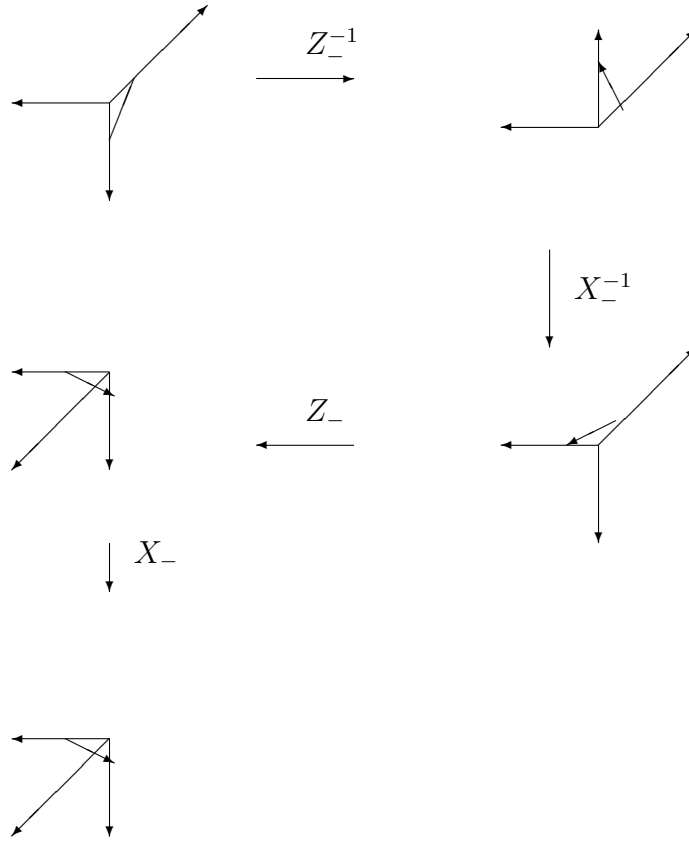
Fig. 1:

$$Z_-^{-1}$$

$$X_-^{-1}$$

$$Z_-$$

$$X_-$$
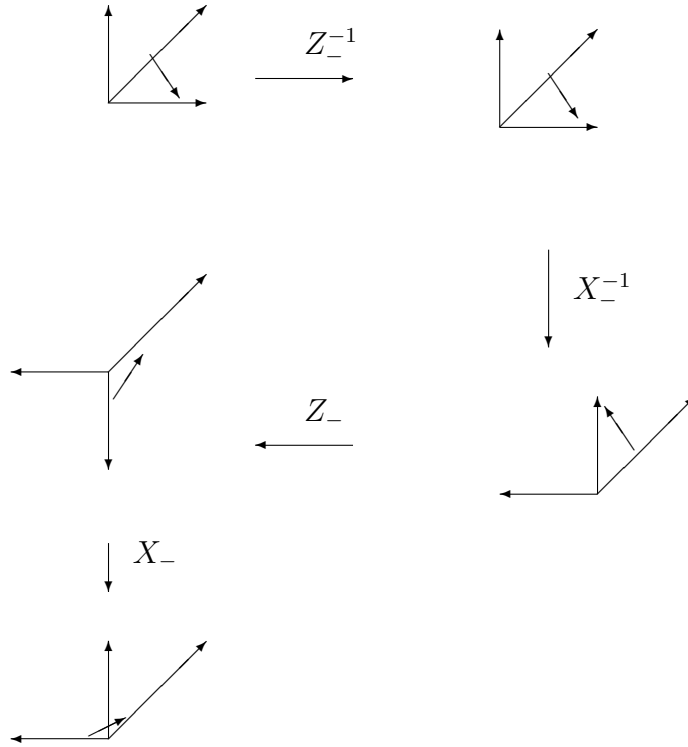
Therefore $(Z_-^{-1}X_-^{-1}Z_-X_-)^2$ sends the up, right, back corner to itself but NOT by the identity. Looking down the interior diagonal from that corner, $(Z_-^{-1}X_-^{-1}Z_-X_-)^2$ rotates the three faces at that corner anti-clockwise through $2\pi/3$. Similarly, $(Z_-^{-1}X_-^{-1}Z_-X_-)^2$ sends the up, right, front corner to itself but NOT by the identity. Looking along the interior diagonal from that corner, $(Z_-^{-1}X_-^{-1}Z_-X_-)^2$ rotates the three faces at that corner anti-clockwise through $2\pi/3$.

Therefore $(Z_-^{-1}X_-^{-1}Z_-X_-)^2$ sends the down, right, front corner to itself but NOT by the identity. Looking up the interior diagonal from that corner, $(Z_-^{-1}X_-^{-1}Z_-X_-)^2$ rotates the three faces at that corner clockwise through $2\pi/3$. Similarly, $(Z_-^{-1}X_-^{-1}Z_-X_-)^2$ sends the down, left, front corner to itself but NOT by the identity. Looking along the interior diagonal from that corner, $(Z_-^{-1}X_-^{-1}Z_-X_-)^2$ rotates the three faces at that corner clockwise through $2\pi/3$.

$$Z_-^{-1} X_-^{-1} Z_- X_-$$
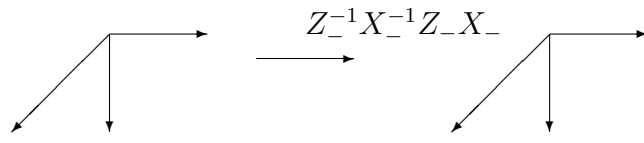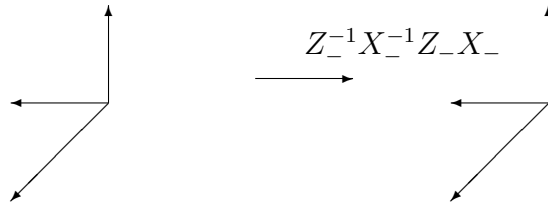
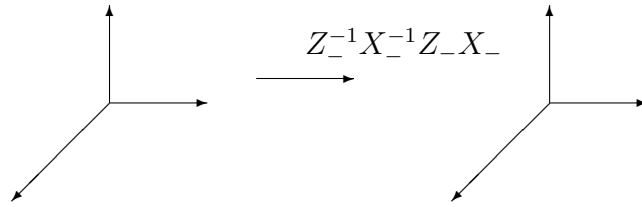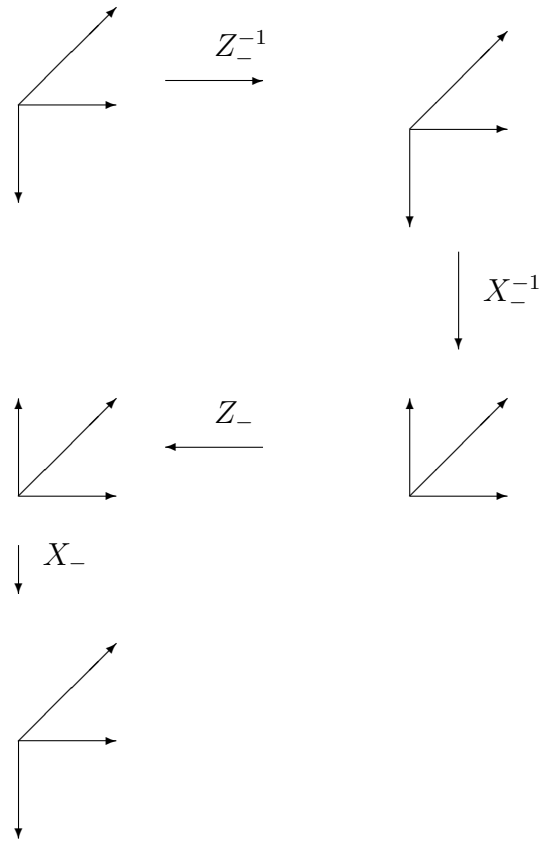$$Z_-^{-1}X_-^{-1}Z_-X_-$$

$$Z_-^{-1} X_-^{-1} Z_- X_-$$

Fig. 8: No change

Therefore, if we hold the cube as shown in the diagram of §1.4 we have up-face (A,B,C,D), down-face $(\alpha, \beta, \gamma, \delta)$, left-face (A,B,$\beta,\alpha$),right-face (D,C, γ, δ), front-face (A, D, δ, α) and back-face (B, C, γ, β). Hence we can label the corner sub-cubes with three letters such as, for example, (d,r,b) for down-right-back. Then the diagrams show us that $Z_-^{-1} X_-^{-1} Z_- X_-$ has the effect of two transpositions

$$(d, l, f) \leftrightarrow (d, r, f) \text{ and } (u, r, f) \leftrightarrow (u, r, b).$$

The standard notation for such a permutation is

$$Z_-^{-1} X_-^{-1} Z_- X_- : \ ((d, l, f), (d, r, f))((u, r, f), (u, r, b)).$$

Similarly, for example, the permutation leaving all corners fixed except

$$(d, r, f) \mapsto (d, r, b) \mapsto (d, l, b)) \mapsto (d, r, f)$$

is denoted by $((d, r, f), (d, r, b), (d, l, b))$.

The following diagram describes the effect of $(Z_-^{-1} X_-^{-1} Z_- X_-)^2$. Namely, $(Z_-^{-1} X_-^{-1} Z_- X_-)^2$ sends each corner to itself, by the identity except for corners C, D, α, δ. At these corners the effect is a clockwise rotation about the inward diagonal if there is a $+$-sign and an anti-clockwise rotation about the inward diagonal if there is a $-$-sign
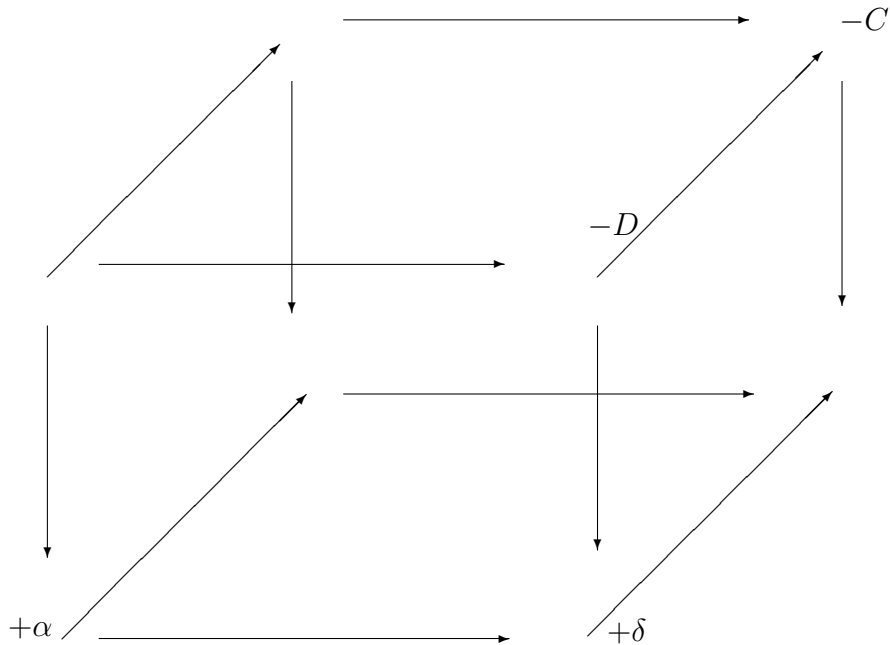
When one of $X_\pm, Y_\pm, Z_\pm$ acts to move one corner to another it sends the inward diagonal of the first corner into the inward diagonal of the second

corner and preserves the clockwise or anti-clockwise nature (i.e. chirality) of rotations about the inward diagonal.

Therefore the effect of conjugating $(Z_-^{-1}X_-^{-1}Z_-X_-)^2$ by $X_+^2 = X_+^{-2}$, which is written $X_+^2(Z_-^{-1}X_-^{-1}Z_-X_-)^2X_+^2$, is depicted as follows:
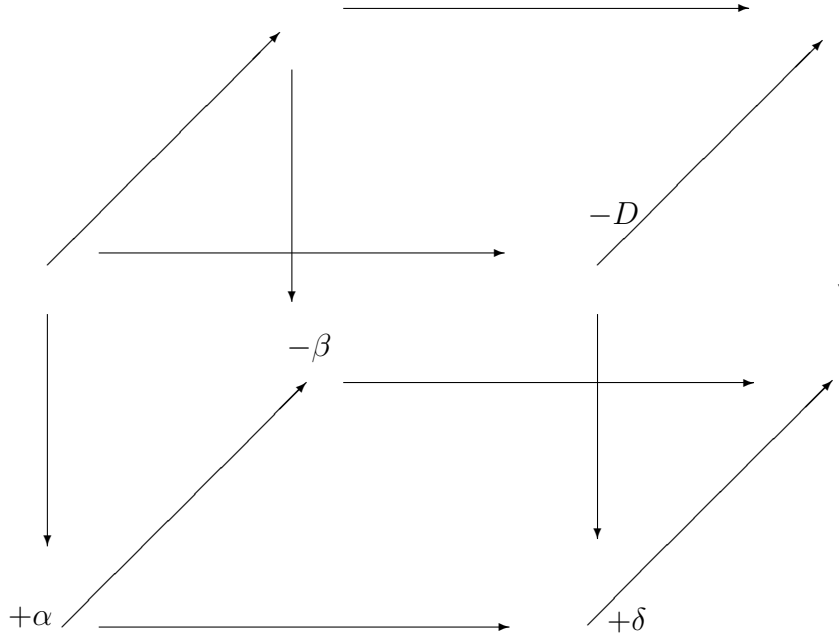
Therefore $X_+^2(Z_-^{-1}X_-^{-1}Z_-X_-)^2X_+^2$ and $(Z_-^{-1}X_-^{-1}Z_-X_-)^2$ have exactly the same effect on all corners except β and C. Therefore combining first $(Z_-^{-1}X_-^{-1}Z_-X_-)^2$ and then the inverse of $X_+^2(Z_-^{-1}X_-^{-1}Z_-X_-)^2X_+^2$ changes only two corners, β and C, in the manner shown below:

$$(Z_-^{-1}X_-^{-1}Z_-X_-)^2 \ X_+^2(Z_-^{-1}X_-^{-1}Z_-X_-)^{-2}X_+^{-2}$$
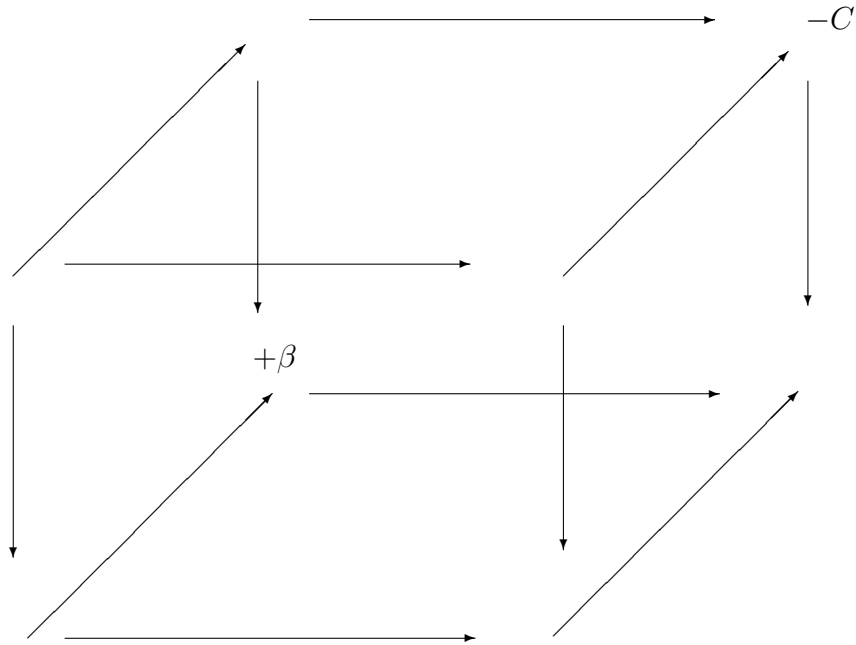
Conjugating this by compositions of $X_\pm, Y_\pm, Z_\pm$ shows that we can obtain, for any pair of corners, a symmetry of the 2×2 Rubik cube which does a $+$-rotation at the first corner and a $-$-rotation at the other, leaving all other corners unchanged. Such symmetries will commute with each other and composing several of them or their inverses we can obtain any symmetry which sends each corner into itself by the identity or a $\pm$-rotation about the inward diagonal at that corner. We can assign $\pm$'s arbitrarly at corners $A, B, C, D, \alpha, \beta$ and γ but then the sign at δ must ensure that the product of all the eight $\pm$'s is a $+$. These symmetries form an elementary abelian 3-subgroup of order 3^7. It is a normal subgroup of the group of all 2×2 Rubik cube symmetries. In fact it is the kernel of the homomorphism of groups which sends a 2×2 Rubik cube symmetry to the permutation of the eight vertices.

If $\mathcal{G}$ denotes the group of all 2×2 Rubik cube symmetries and Σ_8 is the symmetric group of permutations of the eight corners then this homomorphism will be denoted by

$$\Lambda : \mathcal{G} \longrightarrow \Sigma_8.$$

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5. Permutations of corners

5.1. *Permutations of the ordered set of* 8 *objects*

Write

$$A = 1, B = 2, C = 3, D = 4, \alpha = 5, \beta = 6, \gamma = 7, \delta = 8$$

then we think of permutations of the corners are permutations of the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ Re-writing we have

$$X_+ = (B, C, \gamma, \beta) = (2, 3, 7, 6)$$

$$X_- = (A, D, \delta, \alpha) = (1, 4, 8, 5)$$

$$Y_+ = (A, D, C, B) = (1, 4, 3, 2)$$

$$Y_- = (\alpha, \delta, \gamma, \beta) = (5, 8, 7, 6)$$

$$Z_+ = (A, B, \beta, \alpha) = (1, 2, 6, 5)$$

$$Z_- = (D, C, \gamma, \delta) = (4, 3, 7, 8).$$

and, if $X^2 = XX$,

$$X_+^2 = (2, 7)(3, 6)$$

$$X_-^2 = (1, 8)(4, 5)$$

$$Y_+^2 = (1, 3)(4, 2)$$

$$Y_-^2 = (5, 7)(8, 6)$$

$$Z_+^2 = (1, 6)(2, 5)$$

$$Z_-^2 = (4, 7)(3, 8).$$

5.2. *Permutations using matrices*

A matrix is a rectangular array of numbers or mathematical symbols. We are going to depict the effect of $X_\pm, Y_\pm, Z_\pm$ as permutations of the 8 corners in terms of "multiplication" matrices. First we shall depict the set of eight numbers to be permuted by a matrix with one row and eight columns

$$(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8)$$

where the X_i's are the number 1 to 8 and the order they are arrayed in is really important - so this is called an order set of eight numbers. Next we take a matrix with 8 rows and 8 columns to depict the permutation we have in mind. The next bit is tricky. If X_i moves to the j-th position we put a 1 in the j-th position of the i-th row. This recipe will put 8 ones in our matrix

with exactly one in each row or column. Then we fill in all the other numbers by 0's.

Let us make the matrix for X_+ permuting the corners. It sends the 2nd to the 3rd, the 3rd to the 7th, the 7th to the 6th, the 6th to the 2nd and leaves $1, 4, 5, 8$ unmoved.

$$X_+ = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

To perform the "multiplication" of

$$(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8) \times X_+$$

we compute 8 numbers and array then in a row in order where we get the s-th one by adding

$$X_1 \times (1st \text{ number in } s-\text{th column}$$

plus

$$X_2 \times (2nd \text{ number in } s-\text{th column}$$

plus

$$X_3 \times (3rd \text{ number in } s-\text{th column}$$

and so on down to $X_8 \times (8th$ number in s $-$ th column.

This is quite tricky at first but one soon gets the idea of running a finger along the row of X_i's and down the s-th column at the same time! Here we go with our fist example

$$(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8) \times X_+$$

$$= (X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8) \times \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= (X_1, X_6, X_2, X_4, X_5, X_7, X_3, X_8)$$

You can check that the six matrices we need are given by:

$$
X_+ = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad
X_- = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
$$

$$
Y_+ = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad
Y_- = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
$$

$$
Z_+ = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad
Z_- = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

Next we calculate that

$$
Z_- Y_+ X_+ = (1,4,3,6,2)(7,8).
$$

This calculation goes as follows:

$$(1, 2, 3, 4, 5, 6, 7, 8)$$

$$\downarrow \ Z_-$$

$$(1, 2, 4, 8, 5, 6, 3, 7)$$

$$\downarrow \ Y_+$$

$$(2, 4, 8, 1, 5, 6, 3, 7)$$

$$\downarrow \ X_+$$

$$(2, 6, 4, 1, 5, 3, 8, 7)$$

which switches the 7th and the 8th and sends the 1st to the 4th place, the 4th to the 3rd place, the 3rd to the 6th place, the 6th to the 2nd place, as required.

Therefore if we repeat this permuttion five times we find that $(Z_-Y_+X_+)^5$ is the transposition which switches the 7th and 8th

$$(Z_-Y_+X_+)^5 = (7, 8).$$

6. The order of $\mathcal{G}$

6.1. In order to calculate the order $|\mathcal{G}|$ it suffices to calculate the order of the image of Λ since the kernel of Λ has order 3^7. As permutations of the 8-tuple $(A, B, C, D, \alpha, \beta, \gamma, \delta)$ we have

$$X_+ = (B, C, \gamma, \beta)$$

$$X_- = (A, D, \delta, \alpha)$$

$$Y_+ = (A, D, C, B)$$

$$Y_- = (\alpha, \delta, \gamma, \beta)$$

$$Z_+ = (A, B, \beta, \alpha)$$

$$Z_- = (D, C, \gamma, \delta).$$

Each of the above is a 4-cycle and, for example, the notation signifies that X_+ sends B to C, C to γ, γ to β, β to B and fixes A, D, α and δ.

Theorem 6.2.

(i) The order of the subgrop of Σ_8 generated by $\Lambda(X_\pm), \Lambda(Y_\pm), \Lambda(Z_\pm)$ is $40320 = 2^7 \cdot 3^2 \cdot 5 \cdot 7 = 8! = |\Sigma_8|$.

(ii) The order of $\mathcal{G}$ is given by
$$|\mathcal{G}| = 2^7 \cdot 3^9 \cdot 5 \cdot 7.$$

Proof

Part (i) is a calculation first performed for me by Francis Clarke, using the Sage mathematical software package, taking 0.31 seconds! However we have seen in the previous section that $(Z_-Y_+X_+)^5 = (7,8)$ as a permutation of the corners. It is easy to see that one can get any other transposition of corners from this one, by holding the Rubik cube in the right orientation. Therefore the image of Λ is a subgroup of Σ_8 containing all the transpositions. It is well-known in group theory that every element of Σ_8 is a composition of transpositions so the image of Λ is all of Σ_8.

Part (ii) follows immediately from the discussion of §4 and §6.1. This was first verified for me by Graham Gerrard using Sage. □

7. Actually solving the 2×2 Rubik cube

7.1. *Positioning the corners*

Using moves which transpose two corners and leave the remaining corners where they were (possibly twisted) we may put all the corners into their rightful positions (possibly twisted). Here are a list of seven moves which switch corner $8 = \delta$ with the other seven corners.

moves	transposition by numbers	transposition by letters
$(Z_-Y_+X_+)^5$	(γ, δ)	$(7,8)$
$X_+^{-1}(Z_-Y_+X_+)^5 X_+$	(β, δ)	$(6,8)$
$Z_+^{-1}X_+^{-1}(Z_-Y_+X_+)^5 X_+ Z_+$	(α, δ)	$(5,8)$
$Y_+^2 X_+^2 (Z_-Y_+X_+)^5 X_+^2 Y_+^2$	(D, δ)	$(4,8)$
$Y_+ X_+^2 (Z_-Y_+X_+)^5 X_+^2 Y_+^{-1}$	(C, δ)	$(3,8)$
$X_+^2 (Z_-Y_+X_+)^5 X_+^2$	(B, δ)	$(2,8)$
$Y_+^{-1} X_+^2 (Z_-Y_+X_+)^5 X_+^2 Y_+$	(A, δ)	$(1,8)$

7.2. *Rectifying the corners*

Once each corner is where is belongs then it only needs to be rotated. By orientating the cube correctly one of the two corner-twisting examples of §4 will apply to the corner you want to re-orientate. Sort out each corner in turn and you are done!