ON THE MOTIVIC SPECTRA REPRESENTING ALGEBRAIC COBORDISM AND ALGEBRAIC K-THEORY

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Abstract. We show that the motivic spectrum representing algebraic K-theory is a localization of the suspension spectrum of $\mathbb{P}^\infty$, and similarly that the motivic spectrum representing periodic algebraic cobordism is a localization of the suspension spectrum of BGL. In particular, working over $\mathbb{C}$ and passing to spaces of $\mathbb{C}$-valued points, we obtain new proofs of the topological versions of these theorems, originally due to the second author. It follows that algebraic K-theory is E_∞ as a motivic spectrum.

1. Introduction

1.1. **Background and motivation.** The following type of non-connective spectrum was introduced in [18]. Let (X, μ) be a commutative monoid in the homotopy category of pointed spaces let and $\beta \in \pi_n(\Sigma^\infty X)$ be an element in the stable homotopy of X. Now $\Sigma^\infty X$ is a homotopy commutative ring spectrum, and we may invert the "multiplication by β" map

$$\beta_* : \Sigma^\infty X \simeq \Sigma^\infty S^0 \wedge \Sigma^\infty X \xrightarrow{\Sigma^{-n}\beta\wedge 1} \Sigma^{-n}\Sigma^\infty X \wedge \Sigma^\infty X \xrightarrow{\Sigma^{-n}\Sigma^\infty \mu} \Sigma^{-n}\Sigma^\infty X.$$

to obtain a ring spectrum

$$\Sigma^\infty X[1/\beta] := \operatorname{colim}\{\Sigma^\infty X \xrightarrow{\beta_*} \Sigma^{-n}\Sigma^\infty X \xrightarrow{\Sigma^{-n}\beta_*} \Sigma^{-2n}\Sigma^\infty X \longrightarrow \cdots\}$$

with the property that $\beta_* : \Sigma^\infty X[1/\beta] \to \Sigma^{-n}\Sigma^\infty X[1/\beta]$ is an equivalence. In fact, as is well-known, $\Sigma^\infty X[1/\beta]$ is universal among $\Sigma^\infty X$-algebras A in which β becomes a unit.

It was originally shown in [18] (see also [19] for a simpler proof) that the ring spectra $\Sigma^\infty_+ BU[1/\beta]$ and $\Sigma^\infty_+\mathbb{CP}^\infty[1/\beta]$, obtained as above by taking X to be BU_+ or $\mathbb{P}^\infty_+$ and β a generator of $\pi_2 X$ (a copy of $\mathbb{Z}$ in both cases), represent periodic complex cobordism and topological K-theory, respectively. This motivated an attempt in [18] to define algebraic cobordism by replacing $BGL(\mathbb{C})$ in this construction with Quillen's algebraic K-theory spaces [17]. The result was an algebraic cobordism theory, defined in the ordinary stable homotopy category, which was far too large.

By analogy with topological complex cobordism, algebraic cobordism ought to be the universal oriented algebraic cohomology theory. However, there are at least two algebraic reformulations of the topological theory; as a result, there are at least two distinct notions of algebraic cobordism popular in the literature today. One, due to Levine and Morel [8], [9], constructs a universal "oriented Borel-Moore" cohomology theory Ω by generators and relations in a way reminiscent of the construction of the Lazard ring, and indeed the value of Ω on the point is the Lazard ring. However, Ω is not a generalized motivic cohomology theory in the sense of Morel and Voevodsky [13], so it is not represented by a motivic ring spectrum.

The other notion, and the one relevant to this paper, is Voevodsky's spectrum MGL [23]. It is a bona fide motivic cohomology theory in the sense that it is defined directly on the level of motivic spectra. Although the coefficient ring of MGL is still not known (at least in all cases), the orientability of MGL implies that it is an algebra over the Lazard ring, as it carries a formal group law. Provided one defines an orientation as a compatible family of Thom classes for vector bundles, it is immediate that MGL represents the universal oriented motivic cohomology theory; moreover, as shown in [15], and just as in the classical case, the splitting principle implies that it is enough to specify a Thom class for the universal line bundle.

The infinite Grassmannian

$$BGL_n \simeq \mathrm{Grass}_{n,\infty} := \operatorname{colim}_k \mathrm{Grass}_{n,k}$$

represents, in the $\mathbb{A}^1$-local homotopy category , the functor which associates to a variety X the set of isomorphism classes of rank n vector bundles on X. In particular, tensor product of line bundles and Whitney sum of stable vector bundles endow $\mathbb{P}^\infty \simeq BGL_1$ and $BGL \simeq \operatorname{colim}_n BGL_n$ with the structure of abelian group objects in the $\mathbb{A}^1$-homotopy category. Note that, over $\mathbb{C}$, the spaces $\mathbb{P}^\infty(\mathbb{C})$ and $BGL(\mathbb{C})$ underlying the associated complex-analytic varieties are equivalent to the usual classifying spaces $\mathbb{CP}^\infty$ and BU.

We might therefore hypothesize, by analogy with topology, that there are equivalences of motivic ring spectra

$$\Sigma^\infty_+ BGL[1/\beta] \longrightarrow \bigvee_{n\in\mathbb{Z}} \Sigma^{2n,n} MGL \qquad \text{and} \qquad \Sigma^\infty_+ \mathbb{P}^\infty[1/\beta] \longrightarrow K.$$

The purpose of this paper is to prove this hypothesis. In fact, it holds over an arbitrary Noetherian base scheme S of finite Krull dimension, provided one interprets K properly: the Thomason-Troubough K-theory of schemes [22] is not homotopy invariant, and so it cannot possibly define a motivic cohomology theory; rather, the motivic analogue of K-theory is Weibel's homotopy K-theory [27], which agrees with the Thomason-Troubough when restricted to *regular* schemes.

1.2. **Organization of the paper.** We begin with an overview of the theory of oriented motivic ring spectra. The notion of an orientation is a powerful one, allowing us to compute first the oriented cohomology of flag varieties and Grassmannians. We use our calculations to identify the primitive elements in the Hopf algebra $R^0(\mathbb{Z}\times BGL)$ with $R^0(BGL_1)$, a key point in our analysis of the abelian group $R^0(K)$ of spectrum maps from K to R.

The next section is devoted to algebraic cobordism, in particular the proof that algebraic cobordism is represented by the motivic spectrum $\Sigma^\infty_+ BGL[1/\beta]$. We recall the construction of MGL as well as its periodic version $PMGL$ and note the functors they (co)represent as monoids in the homotopy category of motivic spectra. We show that $PMGL$ is becomes equivalent to $\bigvee_n \Sigma^\infty MGL_n[1/\beta]$ and use the isomorphism $R^0(BGL) \cong \prod_n R^0(MGL_n)$ to identify the functors $\mathrm{Rings}(\Sigma^\infty_+ BGL[1/\beta], -)$ and $\mathrm{Rings}(\bigvee_n \Sigma^\infty MGL_n[1/\beta], -)$.

The final section provides the proof that algebraic K-theory is represented by the motivic spectrum $\Sigma^\infty_+\mathbb{P}^\infty[1/\beta]$. First we construct a map; to see that it's an equivalence, we note that it's enough to show that the induced map $R^0(K)\to R^0(\Sigma^\infty_+\mathbb{P}^\infty[1/\beta])$ is an isomorphism for any $PMGL$-algebra R. An element of $R^0(K)$ amounts to a homotopy class of an infinite loop map $\mathbb{Z}\times BGL \simeq \Omega^\infty K \to \Omega^\infty R$; since loop maps $\mathbb{Z}\times BGL \to \Omega^\infty R$ are necessarily additive, we are reduced to looking at maps $\mathbb{P}^\infty \to \Omega^\infty R$. We use this to show that the spaces $\mathrm{map}(K,R)$ and $\mathrm{map}(\Sigma^\infty_+\mathbb{P}^\infty[1/\beta], R)$ both arise as the homotopy inverse limit of the tower associated to the endomorphism of the space $\mathrm{map}(\Sigma^\infty_+\mathbb{P}^\infty, R)$ induced by the action of the Bott map $\mathbb{P}^1\wedge\mathbb{P}^\infty \to \mathbb{P}^\infty$, and are therefore homotopy equivalent.

1.3. **Applications.** Our theorems have a number of useful applications which we hope to pursue in subsequent papers. For now, we sketch two of the most important.

First, it follows immediately from our theorems that both $PMGL$ and K are E_∞ *as motivic spectra*. An E_∞ motivic spectrum is a *coherently* commutative object in an appropriate symmetric monoidal model category of structured motivic spectra, such as P. Hu's motivic $\mathbb{S}$-modules [5] or J.F. Jardine's motivic symmetric spectra [6]; in particular, it is a much stronger condition than merely representing a presheaf of (ordinary) E_∞ spectra on an appropriate site. While this is already known to be the case for algebraic cobordism, where it is clear from the construction of MGL, it is not yet (to the author's knowledge) known for algebraic K-theory, as the topological arguments don't seem to immediately generalize.

Another application, relying on the first, is to the version of derived algebraic geometry that uses E_∞ motivic spectra as its basic building blocks. In [10], J. Lurie shows that $\operatorname{spec}\Sigma^\infty_+\mathbb{P}^\infty[1/\beta]$ is the initial derived scheme over which the derived multiplicative group $\mathbb{G}_R := \operatorname{spec} R \wedge \Sigma^\infty_+\mathbb{Z}$ acquires an "orientation", in the sense that the formal group of $\mathbb{G}_R$ may be identified with the formal spectrum $\mathbb{P}^\infty \otimes \operatorname{spec} R$. Since $\Sigma^\infty_+\mathbb{CP}^\infty[1/\beta]$ represents topological K-theory, this is really a theorem about the relation between K-theory and the derived multiplicative group, and is the starting point for Lurie's program to similarly relate topological modular forms and derived elliptic curves. Hence the motivic version of this theorem may be regarded as a step towards algebraic elliptic cohomology.

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2. Oriented Cohomology Theories

2.1. **Motivic spaces.** Throughout this paper, we write S for a Noetherian base scheme of finite Krull dimension.

Definition 2.1. A *motivic space* is a simplicial sheaf on the Nisnevich site of smooth schemes over S.

We often write 0 for the initial motivic space $\emptyset$, the simplicial sheaf with constant value the set with zero elements, and 1 for the final motivic space S, the simplicial sheaf with constant value the set with one element.

We assume that the reader is familiar with the Morel-Voevodsky $\mathbb{A}^1$-local model structure on the category of motivic spaces used to define the unstable motivic homotopy category [13]. We adhere to this treatment with one exception: we adopt a different convention for indexing the simplicial and algebraic spheres. The *simplicial circle* is the pair associated to the constant simplicial sheaves

$$S^{1,0} := (\Delta^1, \partial\Delta^1);$$

its smash powers are the *simplicial spheres*

$$S^{n,0} := (\Delta^n, \partial\Delta^n).$$

The *algebraic circle* is the multiplicative group scheme $\mathbb{G} := \mathbb{G}_m := \mathbb{A}^1 - \mathbb{A}^0$, pointed by the identity section $1 \to \mathbb{G}$; its smash powers define the *algebraic spheres*

$$S^{0,n} := (\mathbb{G}, 1)^{\wedge n}.$$

Putting the two together, we obtain a bi-indexed family of spheres

$$S^{p,q} := S^{p,0} \wedge S^{0,q}.$$

It is straightforward to show that

$$(\mathbb{A}^n - \mathbb{A}^0, 1) \simeq S^{n-1,n}$$

and

$$(\mathbb{A}^n, \mathbb{A}^n - \mathbb{A}^0) \simeq (\mathbb{P}^n, \mathbb{P}^{n-1}) \simeq S^{n,n}.$$

We emphasize that, according to the more usual convention regarding grading, $S^{p,q}$ is written $S^{p+q,q}$; we find it more intuitive to separate the simplicial and algebraic spheres notationally. Moreover, for this purposes of this paper, the diagonal spheres

$$S^{n,n} \simeq (\mathbb{A}^n, \mathbb{A}^n - \mathbb{A}^0) \simeq (\mathbb{P}^n/, PP^{n-1})$$

are far and away the most important, so they will be abbreviated

$$S^n := S^{n,n}.$$

This allows us to get by with just a single index most of the time.

We extend this convention to suspension and loop functors. That is, $\Sigma(-)$ denotes the endofunctor on pointed motivic spaces (or spectra) defined by

$$\Sigma X := S^1 \wedge X := S^{1,1} \wedge X.$$

Similarly, its right adjoint $\Omega(-)$ is defined by

$$\Omega X := \mathrm{map}_+(S^1, X) := \mathrm{map}_+(S^{1,1}, X).$$

Note that Σ is therefore *not* the categorical suspension, which is to say that the cofiber of the unique map $X \to 1$ is given by $S^{1,0} \wedge X$ instead of $S^{1,1} \wedge X = \Sigma X$. While this may be confusing at first, we feel that the notational simplification that results makes it worthwhile in the end.

2.2. **Motivic spectra.** To form the stable motivic category, we formally add desuspensions with respect to the diagonal spheres $S^n = S^{n,n} = (\mathbb{A}^n, \mathbb{A}^n - \mathbb{A}^0)$.

Definition 2.2. A *motivic prespectrum* is a sequence of pointed motivic spaces

$$\{X(0), X(1), \ldots\},$$

equipped with maps $\Sigma^p X(q) \to X(p+q)$, such that the resulting squares

$$\begin{tikzcd} \Sigma^p \Sigma^q X(r) \arrow[r] \arrow[d] & \Sigma^{p+q} X(r) \arrow[d] \\ \Sigma^p X(q+r) \arrow[r] & X(p+q+r) \end{tikzcd}$$

commute.

Definition 2.3. A motivic prespectrum is a *motivic spectrum* if, for all natural numbers p, q, the adjoints $X(q) \to \Omega^p X(p+q)$ of the prespectrum structure maps $\Sigma^p X(q) \to X(p+q)$ are weak equivalences.

A pointed motivic space $X = (X, 1)$ gives rise to the suspension spectrum $\Sigma^\infty X$, the spectrum associated to the prespectrum with

$$(\Sigma^\infty X)(p) := \Sigma^p X$$

and structure maps

$$\Sigma^q \Sigma^p X \longrightarrow \Sigma^{p+q} X.$$

If X isn't already pointed, we usually write $\Sigma^\infty_+ X$ for $\Sigma^\infty X_+$, where X_+ is the pointed space $(X_+, 1) \simeq (X, 0)$. If X happens to be the terminal object 1, we write $\mathbb{S} := \Sigma^\infty_+ 1$ for the resulting suspension spectrum, the motivic sphere.

We will need that the homotopy category of motivic spectra is closed symmetric monoidal with respect to the smash product. However, we do not focus on the details of its construction, save to say that either P. Hu's theory of motivic $\mathbb{S}$-modules [5] or J.F. Jardine's motivic symmetric spectra [6] will do.

The category of motivic spectra is also enriched over the homotopy category of spaces (or spectra). That is, given a pair of motivic spectra X and Y, there's a space $\mathrm{map}(X,Y)$ of maps between these spectra, defined by

$$\mathrm{map}(X,Y) \simeq \mathrm{holim}_n\, \mathrm{map}(\Omega^\infty\Sigma^n X, \Omega^\infty\Sigma^n Y).$$

Here the map

$$\mathrm{map}(\Omega^\infty\Sigma^{n+1}X, \Omega^\infty\Sigma^{n+1}Y) \to \mathrm{map}(\Omega^\infty\Sigma^n X, \Omega^\infty\Sigma^n Y)$$

sends $f \in \mathrm{map}(\Omega^\infty\Sigma^{n+1}X, \Omega^\infty\Sigma^{n+1}Y)$ to the composite

$$\Omega^\infty\Sigma^n X \xrightarrow{\simeq} \Omega^{\infty+1}\Sigma^{n+1}X \xrightarrow{\Omega f} \Omega^{\infty+1}\Sigma^{n+1}Y \xrightarrow{\simeq} \Omega^\infty\Sigma^n Y\ ,$$

where $\Omega^{\infty+1} = \Omega\Omega^\infty \simeq \Omega^\infty\Omega$.

We record the following lemma for future use.

Lemma 2.4. *Suppose given a commuting diagram of spectra*

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "i"'] & Z \arrow[d, "i"] \\ Y \arrow[r, "f"] \arrow[ur] \arrow[d, "r"'] & Y \arrow[d, "r"] \\ Z \arrow[r, "g"] & Z \end{tikzcd}$$

in which the vertical composites are equivalent to the identity. Then the natural map from the homotopy limit of the tower $\{\cdots \to Z \to Z\}$, obtained by iterating g, to the homotopy limit of the tower $\{\cdots \to Y \to Y\}$, obtained by iterating f, is an equivalence.

Proof. Since Z is a retract of Y, we may write $Y \simeq Z+Y-Z$ for some spectrum $Y-Z$; note that, with respect to this equivalence, $f : Y \to Y$ decomposes as $g+0 : Z+Y-Z \to Z+Y-Z$. This gives a commuting diagram

$$\begin{tikzcd} \prod_n Y-Z \arrow[r] \arrow[d] & \prod_n Y-Z \arrow[d] \\ \prod_n Z+Y-Z \arrow[r] \arrow[d] & \prod_n Z+Y-Z \arrow[d] \\ \prod_n Z \arrow[r] & \prod_n Z \end{tikzcd}$$

in which the maps are defined (via their functors of points) as follows: given $\{y_n\} : X \to \prod_n Y-Z$ and $\{z_n\} : X \to \prod_n Z$, the top maps sends $\{y_n\}$ to $\{y_n\}$, the middle map sends $\{z_n+y_n\}$ to $\{z_n-g(z_{n+1})+y_n\}$, and the bottom map sends $\{z_n\}$ to $\{z_n-g(z_{n+1})\}$. Since the fiber of the top map is trivial, we see that the fiber of the middle map is equivalent to the fiber of the bottom map. But the fiber of the bottom map is the homotopy limit of the tower defined by g, and by assumption, $z_n+y_n-f(z_{n+1}+y_{n+1}) \simeq z_n+y_n-g(z_{n+1}) \simeq z_n-g(z_{n+1})+y_n$, so the fiber of the middle map is the homotopy limit of the tower defined by f. $\square$

2.3. **Motivic ring spectra.** In this paper, unless appropriately qualified, a ring spectrum will always mean a commutative monoid in the homotopy category of motivic spectra. We reiterate that a motivic spectrum is a $\mathbb{P}^1$-spectrum, that is, it admits desuspensions by algebraic spheres as well as simplicial spheres.

Definition 2.5. A motivic ring spectrum R is *periodic* if the graded ring π_*R contains a unit in degree one.

Remark 2.6. Since $\pi_1 R$ is by definition $\pi_0 \operatorname{map}_+(\mathbb{P}^1, \Omega^\infty R)$, and over $\operatorname{spec}\mathbb{C}$, $\mathbb{P}^1(\mathbb{C}) \simeq \mathbb{CP}^1$, the topological 2-sphere, this is compatible with the notion of an *even periodic* ring spectrum so common in ordinary stable homotopy theory.

Proposition 2.7. *If R is periodic then $R^0 \simeq R^n$ for all n.*

Proof. If R is periodic then, for any n, the multiplication by β^n map

$$R \longrightarrow \Sigma^{-n} R$$

is an equivalence. In particular, the map

$$R^0 \simeq \Omega^\infty R \simeq \Omega^\infty \Sigma^{-n} R \simeq R^n.$$

on infinite loop spaces is an equivalence. □

If we start with any old motivic ring spectrum R, there is a canonical way to make R periodic. Let $P\mathbb{S}$ denote the periodic sphere, the motivic spectrum

$$P\mathbb{S} := \bigvee_{n\in\mathbb{Z}} \mathbb{S}^n.$$

Then, with respect to the multiplication induced by the equivalences $\mathbb{S}^p \wedge \mathbb{S}^q \to \mathbb{S}^{p+q}$, the unit in degree one given by the inclusion $\mathbb{S}^1 \to P\mathbb{S}$ makes $P\mathbb{S}$ into a periodic $\mathbb{S}$-algebra.

Clearly the homotopy category of periodic ring spectra (motivic or otherwise) is equivalent to the full subcategory of the homotopy category of spectra which admit a ring map from $P\mathbb{S}$. It follows that

$$PR := P\mathbb{S} \underset{\mathbb{S}}{\wedge} R$$

is a periodic ring spectrum equipped with a ring map $R \to PR$.

Proposition 2.8. *Let R be a motivic ring spectrum. Then homotopy classes of ring maps $P\mathbb{S} \to R$ naturally biject with units in $\pi_1 R$.*

Proof. By definition, ring maps $P\mathbb{S} \to R$ are indexed by families of elements $r_n \in \pi_n R$ with $r_m r_n = r_{m+n}$ and $r_0 = 1$. Hence $r_n = r_1^n$, and in particular $r_{-1} = r_1^{-1}$. □

Corollary 2.9. *Let R be a (motivic) ring spectrum and Q a periodic (motivic) ring spectrum. Then the set of homotopy classes of ring maps $PR \to Q$ is naturally isomorphic to the set of pairs consisting of a homotopy class of ring map $R \to Q$ and a unit in $\pi_1 Q$.*

2.4. **Orientations.** Let R be a motivic ring spectrum.

Definition 2.10. The *Thom space* of an n-plane bundle $V \to X$ is the pair $(V, V - X)$, where $V - X$ denotes the complement in V of the zero section $X \to V$.

Given two vector bundles $V \to X$ and $W \to Y$, the Thom space $(V \times W, V \times W - X \times Y)$ of the product bundle $V \times W \to X \times Y$ is equivalent (even isomorphic) to the smash product $(V, V - X) \wedge (W, W - Y)$ of the Thom spaces. Since the Thom space of the trivial 1-dimensional bundle $\mathbb{A}^1 \to \mathbb{A}^0$ is the motivic 1-sphere $S^1 \simeq (\mathbb{A}^1, \mathbb{A}^1 - \mathbb{A}^0)$, we see that the Thom space of the trivial n-dimensional bundle $\mathbb{A}^n \to \mathbb{A}^0$ is the motivic n-sphere $S^n \simeq (\mathbb{A}^n, \mathbb{A}^n - \mathbb{A}^0)$. Note that the complement of the zero section $\mathbb{L} - \mathbb{P}^\infty$ of the universal line bundle $\mathbb{L} \to \mathbb{P}^\infty \simeq B\mathbb{G}$ is equivalent to the total space of the universal principal $\mathbb{G}$-bundle $E\mathbb{G} \to B\mathbb{G}$, which is contractible. Hence the Thom space of $\mathbb{L} \to \mathbb{P}^\infty$ is equivalent to $(\mathbb{P}^\infty, \mathbb{P}^0)$, and the Thom space of the restriction of $\mathbb{L} \to \mathbb{P}^\infty$ along the inclusion $\mathbb{P}^1 \to \mathbb{P}^\infty$ is equivalent to $(\mathbb{P}^1, \mathbb{P}^0) \simeq S^1$.

Definition 2.11. An *orientation* of R is the assignment, to each m-plane bundle $V \to X$, of a class $\theta(V/X) \in R^m(V, V - X)$, in such a way that

(1) for any $f : Y \to X$, the class $\theta(f^*V/Y)$ of the restriction $f^*V \to Y$ of $V \to X$ is equal to the restriction $f^*\theta(V/X)$ of the class $\theta(V/X)$ in $R^m(f^*V, f^*V - Y)$,

(2) for any n-plane bundle $W \to Y$, the (external) product $\theta(V/X) \times \theta(W/Y)$ of the classes $\theta(V/X)$ and $\theta(W/Y)$ is equal to the class $\theta(V \times W/X \times Y)$ of the (external) product of $V \to X$ and $W \to Y$ in $R^{m+n}(V \times W, V \times W - X \times Y)$, and

(3) if $\mathbb{L} \to \mathbb{P}^\infty$ is the universal line bundle and $i : \mathbb{P}^1 \to \mathbb{P}^\infty$ denotes the inclusion, then $i^*\theta(\mathbb{L}/\mathbb{P}^\infty) \in R^1(f^*\mathbb{L}, f^*\mathbb{L} - \mathbb{P}^1)$ corresponds to $1 \in R^0(S^0)$ via the isomorphism $R^0(S^0) \cong R^1(S^1) \cong R^1(f^*\mathbb{L}, f^*\mathbb{L} - \mathbb{P}^1)$.

Given an orientation of R, the class $\theta(V/X) \in R^n(V, V - X)$ associated to a n-plane bundle $V \to X$ is called the *Thom class* of $V \to X$. The main utility of Thom classes is that they define $R^*(X)$-module isomorphisms $R^*(X) \to R^{*+n}(V, V - X)$.

Remark 2.12. The naturality condition implies that it is enough to specify Thom classes for the universal vector bundles $V_n \to BGL_n$. We write MGL_n for the Thom space of $V_n \to BGL_n$ and θ_n for $\theta(V_n/BGL_n) \in R^n(MGL_n)$.

2.5. **Basic calculations in oriented cohomology.** In this section we fix an oriented motivic ring spectrum R equipped with a unit $\mu \in R^{-1}$. Note that we can use μ to move the Thom classes $\theta_n \in R^n(MGL_n)$ to degree zero Thom classes $\vartheta_n := \mu^n\theta_n \in R^0(MGL_n)$. The following calculations are well known.

Proposition 2.13. *The first Chern class of the tautological line bundle on $\mathbb{P}^n$ defines a ring isomorphism $R^0[\lambda]/(\lambda^{n+1}) \to R^0(\mathbb{P}^n)$.*

Proof. An orientation on R gives rise to a theory of Chern classes in the usual way (see [1] or [15], for example). Inductively, one has a morphism of exact sequences

$$\begin{array}{ccccc} \lambda^n R^0[\lambda]/(\lambda^{n+1}) & \longrightarrow & R^0[\lambda]/(\lambda^{n+1}) & \longrightarrow & R^0[\lambda]/(\lambda^n) \\ \downarrow & & \downarrow & & \downarrow \\ R^0(\mathbb{P}^n, \mathbb{P}^{n-1}) & \longrightarrow & R^0(\mathbb{P}^n) & \longrightarrow & R^0(\mathbb{P}^{n-1}) \end{array}$$

in which the left and right, and hence also the middle, vertical maps are isomorphisms. □

Proposition 2.14. *The first Chern class of the tautological line bundle on $\mathbb{P}^\infty$ defines a ring isomorphism $R^0[\![\lambda]\!] \cong \lim R^0[\lambda]/(\lambda^n) \to R^0(\mathbb{P}^\infty_+)$.*

Proof. We must check that the $\lim^1$ term in the exact sequence

$$0 \longrightarrow \lim{}^1 R^{-1,0}(\mathbb{P}^n) \longrightarrow R^0(\mathbb{P}^\infty) \longrightarrow \lim R^0(\mathbb{P}^n)$$

vanishes, but once again, the maps

$$R^{-1,0}(\mathbb{P}^n) \longrightarrow R^{-1,0}(\mathbb{P}^{n-1})$$

are surjective. □

Corollary 2.15. *For each n, the natural map*

$$R_0(\mathbb{P}^n) \longrightarrow \hom(R^0(\mathbb{P}^n), R^0)$$

is an isomorphism.

Proof. The dual argument of shows that $R_0(\mathbb{P}^n)$ is free of rank $n+1$ over R_0. □

The following proposition is straight out of Atiyah [4].

Proposition 2.16. *Let $p : Y \to X$ be a map of quasicompact S-schemes such and suppose given homogeneous elements $y_1, \dots, y_n \in R^0(Y)$. Let M be the free abelian group on the $y_1, \dots, y_n$, and suppose that X has a cover by open subschemes U such that for all open V in U, the natural map*

$$R^0(V) \otimes M \longrightarrow R^0(p^{-1}V)$$

is an isomorphism. Then, for any open W in X, the map

$$R^0(X, W) \otimes M \longrightarrow R^0(Y, p^{-1}W)$$

is an isomorphism.

Proof. An open subscheme U of X is said to be *good* if, for any open V in U,

$$R^0(V) \otimes M \cong R^0(p^{-1}V). \tag{2.17}$$

As tensoring with M preserves exact sequences, the long exact sequence of the pair (U, V) induces an isomorphism

$$R^0(U, V) \otimes M \to R^0(p^{-1}U, p^{-1}V). \tag{2.18}$$

Hence X is locally good; since X is also quasicompact, we may deduce it's goodness inductively by showing that $U_1 \cup U_2$ is good whenever U_1 and U_2 are good. But any open subscheme of $U_1 \cup U_2$ decomposes as a union $V_1 \cup V_2$ with $V_i = V \cap U_i$. Since V_2 is good and $V/V_1 \cong V_2/V_1 \cap V_2$, we see that 2.18 holds with (U, V) replaced by (V, V_1); since V_1 is good, the long exact sequence for the pair (V, V_1) shows that 2.17 holds. □

Corollary 2.19. *Let $p : V \to X$ be a vector bundle of rank n and let $L \to \mathbb{P}(V)$ be the tautological line bundle. Then the map which sends λ to the first Chern class of L induces an isomorphism*

$$R^0(X)[\lambda]/(\lambda^n - \lambda^{n-1}c_1V + \cdots + (-1)^n c_n V) \to R^0(\mathbb{P}(V))$$

of R^0-algebras.

Proof. We must show that $R^0(X)[\lambda] \to R^0(\mathbb{P}(V))$ is surjective with kernel the ideal generated by $\sum(-1)^i\lambda^i c_i V$. Surjectivity follows from previous Proposition 2.16, which implies that $R^0(\mathbb{P}(V)$ is free of rank n over $R^0(X)$, as the cohomology of the fiber $\mathbb{P}(\mathbb{A}^n)$ is a free R^0-module of rank n. Writing $q : \mathbb{P}(V) \to X$ for the projection, the relation $q^*V = L \oplus W$ implies that $\Sigma(-1)^i\lambda^i c_i V$ is in the kernel of this map, giving a map $R^0(X)[\lambda]/(\Sigma(-1)^i\lambda^i c_i V) \to R^0(\mathbb{P}(V))$. Since this is a surjective map of finite free R^0-modules, it must be an isomorphism. □

Proposition 2.20. *Let $V \to X$ be a rank n vector bundle,* $\mathrm{Flag}(V) \to X$ *the associated flag bundle, and $\sigma_k(x_1, \dots, x_n)$, $1 \le k \le n$, the k^{th} elementary symmetric function in the indeterminates λ_i. Then the map*

$$R^0(X)[\lambda_1, \dots, \lambda_n]/(\{c_k(V) - \sigma_k(\lambda_1, \dots, \lambda_n)\}_{k>0}) \longrightarrow R^0(\mathrm{Flag}(V))$$

which sends the λ_i to the first Chern classes of the n tautological line bundles on $\mathrm{Flag}(V)$*, is an isomorphism.*

Proof. The relations among the Chern classes imply that the map is well-defined. By 2.16, it follows inductively from the fibrations $\mathrm{Flag}(\mathbb{A}^{n-1}) \to \mathrm{Flag}(\mathbb{A}^n) \to \mathbb{P}^{n-1}$ that $R^0(\mathrm{Flag}(\mathbb{A}^n))$ is free of rank $n!$ over R^0. Thus, again using 2.16, $R^0(\mathrm{Flag}(V))$ is free of rank $n!$ over $R^0(X)$. As $R^0(X)[\lambda_1, \dots, \lambda_n]/(\{c_k(V) - \sigma_k(\lambda_1, \dots, \lambda_n)\}_{k>0})$ is also free of rank $n!$ over $R^0(X)$, the map must be an isomorphism. □

Proposition 2.21. *The natural map*

$$R^0(BGL_n) \longrightarrow R^0(BGL_1^n)^{\Sigma_n}$$

is an isomorphism.

Proof. Writing $V_n \to BGL_n$ for the tautological vector bundle, we have an equivalence $\mathrm{Flag}(V_n) \simeq BGL_1^n$. Inductively, we have isomorphisms

$$R^0[\![\lambda_1, \dots, \lambda_n]\!] \longrightarrow R^0(BGL_1^n)$$

and the map

$$R^0(BGL_n) \longrightarrow R^0(BGL_1^n) \cong R^0[\![\lambda_1, \dots, \lambda_n]\!]$$

factors through the invariant subring $R^0[\![\lambda_1, \dots, \lambda_n]\!]^{\Sigma_n}$. Since $R^0[\![\lambda_1, \dots, \lambda_n]\!]$ is free of rank $n!$ over $R^0[\![\lambda_1, \dots, \lambda_n]\!]^{\Sigma_n}$, it follows that the map $R^0(BGL_n) \to R^0(BGL_1^n)^{\Sigma_n}$ is an isomorphism. □

Corollary 2.22. *The natural map*

$$R^0(BGL_n) \longrightarrow \hom(\mathrm{Sym}^n_{R_0} R_0(\mathbb{P}^\infty), R_0)$$

is an isomorphism.

Proof. By the previous Proposition 2.21, we need only check this for $n = 1$. But $R^0(\mathbb{P}^m) \cong \hom(R_0(\mathbb{P}^m), R_0)$, both being free of rank $m + 1$ over R^0, and

$$R^0(\mathbb{P}^\infty) \cong \lim R^0(\mathbb{P}^m) \cong \hom(\mathrm{colim}\, R_0(\mathbb{P}^m), R_0) \cong \hom(R_0(\mathbb{P}^\infty), R_0)$$

by Proposition 2.5. □

Corollary 2.23. *There are isomorphisms $R^0(BGL) \cong \lim_n R^0(BGL_n) \cong R^0[\![c_1, c_2, \dots]\!]$.*

Proof. The $\lim^1$ term in the short exact sequence

$$0 \longrightarrow \lim\nolimits^1_n R^{1,0}(BGL_n) \longrightarrow R^0(BGL) \longrightarrow \lim\nolimits_n R^0(BGL_n)$$

vanishes since the maps $R^{1,0}(BGL_n) \to R^{1,0}(BGL_{n-1})$ are surjective. □

2.6. **Primitives in the oriented cohomology of BGL.** Let R be an oriented ring spectrum. The group completion $BGL_{\mathbb{Z}}$ (usually written $\mathbb{Z} \times BGL$) of the additive monoid $BGL_{\mathbb{N}} = \coprod_{n\in\mathbb{N}} BGL_n$ fits into a fibration sequence

$$BGL \longrightarrow BGL_{\mathbb{Z}} \longrightarrow \mathbb{Z}.$$

In particular, it is a commutative monoid in the motivic homotopy category.

Lemma 2.24. *Let $\mathrm{Add}(BGL_{\mathbb{Z}}, \Omega^\infty R)$ denote the abelian group of homotopy classes of additive maps $BGL_{\mathbb{Z}} \to \Omega^\infty R$. Then the inclusion*

$$\mathrm{Add}(BGL_{\mathbb{Z}}, \Omega^\infty R) \longrightarrow R^0(BGL_{\mathbb{Z}})$$

identifies $\mathrm{Add}(BGL_{\mathbb{Z}}, \Omega^\infty R)$ with the abelian group of primitive elements in the Hopf algebra $R^0(BGL_{\mathbb{Z}})$.

Proof. By definition, there is an equalizer diagram

$$\mathrm{Add}(BGL_{\mathbb{Z}}, \Omega^\infty R) \longrightarrow R^0(BGL_{\mathbb{Z}}) \rightrightarrows (BGL_{\mathbb{Z}} \times BGL_{\mathbb{Z}})$$

associated to the square

$$\begin{tikzcd} BGL_{\mathbb{Z}} \times BGL_{\mathbb{Z}} \arrow[r] \arrow[d] & BGL_{\mathbb{Z}} \arrow[d] \\ \Omega^\infty R \times \Omega^\infty R \arrow[r] & \Omega^\infty R \end{tikzcd}$$

in which the horizontal maps are the addition maps. Let δ denote the Hopf algebra diagonal

$$\delta : R^0(BGL_{\mathbb{Z}}) \longrightarrow R^0(BGL_{\mathbb{Z}} \times BGL_{\mathbb{Z}}) \cong R^0(BGL_{\mathbb{Z}}) \underset{R^0}{\widehat{\otimes}} R^0(BGL_{\mathbb{Z}}).$$

Then the equalizer consists of those $f \in R^0(BGL_{\mathbb{Z}})$ such that $\delta(f) = f \otimes 1 + 1 \otimes f$. This identifies $\mathrm{Add}(BGL_{\mathbb{Z}}, \Omega^\infty R)$ with the primitive elements in $R^0(BGL_{\mathbb{Z}})$. □

Lemma 2.25. *There are natural isomorphisms*

$$\mathrm{Add}(BGL_{\mathbb{Z}}, \Omega^\infty R) \cong \mathrm{Add}(BGL, \Omega^\infty R) \times \mathrm{Add}(\mathbb{Z}, \Omega^\infty R) \cong \mathrm{Add}(BGL, \Omega^\infty R) \times R^0.$$

Proof. The product of additive maps is additive, and, in any category with finite products and countable coproducts, $\mathbb{Z} = \coprod_{\mathbb{Z}} 1$ is the free abelian group on the terminal object 1. □

Proposition 2.26. *The map*

$$\mathrm{Add}(BGL_{\mathbb{Z}}, \Omega^\infty R) \longrightarrow R^0(BGL_1),$$

obtained by restricting an additive map $BGL_{\mathbb{Z}} \to \Omega^\infty$ along the inclusion $BGL_1 \to BGL_{\mathbb{Z}}$, is an isomorphism.

Proof. By Lemma 2.25, it's enough to show that the inclusion $(BGL_1, 1) \to (BGL, 1)$ induces an isomorphism

$$\mathrm{Add}(BGL, \Omega^\infty R) \longrightarrow R^0(BGL_1, 1).$$

Thus let $M = R_0(BGL_1, 1)$, and consider the R_0-algebra

$$A := \bigoplus_{n\geq 0} \mathrm{Sym}^n_{R_0} M$$

together with its augmentation ideal

$$I := \bigoplus_{n>0} \mathrm{Sym}^n_{R_0} M.$$

We have isomorphisms of split short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & R^0(BGL,1) \arrow[r] \arrow[d] & R^0(BGL) \arrow[r] \arrow[d] & R^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \hom_{R_0}(I,R_0) \arrow[r] & \hom_{R_0}(A,R_0) \arrow[r] & \hom_{R_0}(R_0,R_0) \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & R^0(BGL^{\times 2},BGL^{\vee 2}) \arrow[r] \arrow[d] & R^0(BGL^{\times 2}) \arrow[r] \arrow[d] & R^0(BGL^{\vee 2}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \hom_{R_0}(I\otimes_{R_0} I,R_0) \arrow[r] & \hom_{R_0}(A\otimes_{R_0} A,R_0) \arrow[r] & \hom_{R_0}(R_0\oplus I^{\oplus 2},R_0) \arrow[r] & 0
\end{tikzcd}$$

of R^0-modules. According to Lemmas 2.24 and 2.25, we have an exact sequence

$$0 \longrightarrow \mathrm{Add}(BGL,\Omega^\infty) \longrightarrow R^0(BGL) \longrightarrow R^0(BGL^{\times 2},BGL^{\vee 2})$$

in which the map on the right is the cohomology of the map

$$\mu - p_1 - p_2 : (BGL^{\times 2},BGL^{\vee 2}) \longrightarrow (BGL,1)$$

(μ is the addition and the p_i are the two projections); moreover, this map is dual to the multiplication $I\otimes_{R_0} I \to I$. Hence these short exact sequences assemble into a diagram

$$\begin{tikzcd}
0 \arrow[r] & \hom_{R_0}(I/I^2,R_0) \arrow[r] \arrow[d] & \mathrm{Add}(BGL,\Omega^\infty R) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \hom_{R_0}(I,R_0) \arrow[r] \arrow[d] & R^0(BGL) \arrow[r] \arrow[d] & R^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \hom_{R_0}(I\otimes_{R_0} I,R_0) \arrow[r] & R^0(BGL^{\times 2}) \arrow[r] & \hom_{R_0}(R_0\oplus I^{\oplus 2},R_0) \arrow[r] & 0
\end{tikzcd}$$

of short exact sequences by the Snake Lemma. In particular, we see that $\mathrm{Add}(BGL,\Omega^\infty R)$ is naturally identified with the dual $\hom_{R_0}(I/I^2,R_0)$ of the module of indecomposables I/I^2. But $I/I^2 \cong M = R_0(BGL_1,1)$, the duality map

$$R^0(BGL_1,1) \longrightarrow \hom_{R_0}(R_0(BGL_1,1),R_0)$$

is an isomorphism, and the restriction $R^0(BGL,1) \to R^0(BGL_1,1)$ is dual to the inclusion $M \to I$. □

3. Algebraic Cobordism

3.1. **The representing spectrum.** For each natural number n, let $V_n \to BGL_n$ denote the universal n-plane bundle over BGL_n. Then the Thom spaces

$$MGL_n := (V_n, V_n - BGL_n)$$

come equipped with natural maps

$$MGL_p \wedge MGL_q \to MGL_{p+q}$$

defined as the composite of the isomorphism

$$(V_p, V_p - BGL_p) \wedge (V_q, V_q - BGL_q) \longrightarrow (V_p \times V_q, V_p \times V_q - BGL_p \times BGL_q)$$

and the map on Thom spaces associated to the inclusion of vector bundles

$$\begin{tikzcd} V_p \times V_q \arrow[r] \arrow[d] & V_{p+q} \arrow[d] \\ BGL_p \times BGL_q \arrow[r] & BGL_{p+q}. \end{tikzcd}$$

Restricting this map of vector bundles along the inclusion $1 \times BGL_q \to BGL_p \times BGL_q$ gives a map of Thom spaces

$$(\mathbb{A}^p, \mathbb{A}^p - \mathbb{A}^0) \wedge MGL_q \to MGL_{p+q},$$

and these maps comprise the structure maps of the prespectrum MGL. The associated spectrum is defined by

$$MGL(p) := \operatorname{colim}_q \Omega^q MGL_{p+q},$$

as evidently the adjoints

$$MGL(q) \simeq \operatorname{colim}_r \Omega^r MGL_{q+r} \simeq \operatorname{colim}_r \Omega^{p+r} MGL_{p+q+r} \simeq \Omega^p MGL(p+q)$$

of the structure maps $\Sigma^p MGL(q) \to MGL(p+q)$ are equivalences.

Definition 3.1 (Voevodsky [23])**.** Algebraic cobordism is the motivic cohomology theory represented by the motivic spectrum MGL.

3.2. **Algebraic cobordism is the universal oriented motivic ring spectrum.** Just as in ordinary stable homotopy theory, the Thom classes $\theta_n \in R^n(MGL_n)$ coming from an orientation on a motivic ring spectrum R assemble to give a ring map $\theta : MGL \to R$. We begin with a brief review of this correspondence.

Proposition 3.2 (Panin, Pimenov, Röndigs [15])**.** *Let R be a commutative monoid in the homotopy category of motivic spectra. Then the set of monoidal maps $MGL \to R$ is naturally isomorphic to the set of orientations on R.*

Proof. The classical analysis of complex orientations on ring spectra R generalizes immediately. A spectrum map $\theta : MGL \to R$ is determined by a compatible family of maps $\theta_n : MGL_n \to R^n$, which is to say a family of universal Thom classes $\theta_n \in R^n(MGL_n)$. An arbitrary n-plane bundle $V \to X$, represented by a map $X \to BGL_n$, induces a map of Thom spaces $V/V-X \to MGL_n$, so θ_n restricts to a Thom class in $R^n(V/V-X)$. Moreover, these Thom classes are multiplicative and unital precisely when $\theta : MGL \to R$ is monoidal. Similarly, an orientation on R has, as part of its data, Thom classes $\theta_n \in R^n(MGL_n)$ of universal bundles which assemble to form a ring map $\theta : MGL \to R$. □

Again, just as in topology, an orientation on R is equivalent to a compatible family of R-theory Chern classes for vector bundles $V \to X$. This follows from the Thom isomorphism $R^*(BGL_n) \cong R^*(MGL_n)$.

More difficult is the fact that an orientation on a ring spectrum R is uniquely determined by the first Thom class alone; that is, a class $\theta_1 \in R^1(BGL_1) = R$ whose restriction $i^*\theta_1 \in R^1(S^1)$ along the inclusion $S^1 \to MGL_1$ corresponds to $1 \in R^0(S^0)$ via the suspension isomorphism $R^1(S^1) \cong R^0(S^0)$. This is a result of the splitting principle, which allows one to construct Thom classes (or Chern classes) for general vector bundles by descent from a space over which they split. See Adams [1] and Panin-Pimenov-Roendigs [15] for details.

3.3. **A ring spectrum equivalent to $PMGL$.** The wedge

$$\bigvee_{n\in\mathbb{N}} \Sigma^\infty MGL_n$$

forms a ring spectrum with unit $\mathbb{S} \simeq \Sigma^\infty MGL_0$ and multiplication

$$\bigvee_p \Sigma^\infty MGL_p \wedge \bigvee_q \Sigma^\infty MGL_q \longrightarrow \bigvee_{p,q} \Sigma^\infty MGL_p \wedge MGL_q \longrightarrow \bigvee_r \Sigma^\infty MGL_r$$

induced by the maps $MGL_p \wedge MGL_q \to MGL_{p+q}$. Evidently, a (homotopy class of a) ring map $\bigvee_n \Sigma^\infty MGL_n \to R$ is equivalent to a family of *degree zero* Thom classes

$$\vartheta_n \in R^0(MGL_n)$$

with $\vartheta_0 = 1 \in R^0(MGL_0) = R^0$ such that ϑ_{p+q} restricts via $MGL_p \wedge MGL_q \to MGL_{p+q}$ to the product $\vartheta_p\vartheta_q$. This is *not* the same as an orientation on R, as there is nothing forcing $\vartheta_1 \in R^0(MGL_1)$ to restrict to a unit in $R^0(S^1)$. Clearly we should impose this condition, which amounts to inverting $\beta : \mathbb{P}^1 \to \mathbb{P}^\infty$.

Proposition 3.3. *A ring map $PMGL \to R$ induces a ring map $\bigvee_n \Sigma^\infty MGL_n[1/\beta] \to R$.*

Proof. A ring map $\theta : PMGL \to R$ consists of a ring map $MGL \to R$ and a unit $\mu \in R^{-1}$. This specifies Thom classes $\theta_n \in R^n(MGL_n)$, and therefore Thom classes

$$\vartheta_n := \mu^n\theta_n \in R^0(MGL_n)$$

such that

$$\vartheta_p\vartheta_q = \mu^{p+q}\theta_p\theta_q = \mu^{p+q}i^*\theta_{p+q} = i^*\vartheta_{p+q} \in R^0(MGL_p \wedge MGL_q),$$

where i is the map $MGL_p \wedge MGL_q \to MGL_{p+q}$. This gives a ring map $\vartheta : \bigvee_n \Sigma^\infty MGL_n \to R$, and therefore the desired map provided β is sent to a unit. But this is clear: as a class in $R^0(S^1)$,

$$\vartheta(\beta) = \beta^*\vartheta_1 = \mu\beta^*\theta_1,$$

and $\beta^*\theta_1 \in R^1(S^1)$ is the image of $1 \in R^0(S^0)$ under the isomorphism $R^0(S^0) \cong R^1(S^1)$. □

Proposition 3.4. *The ring map $\bigvee_{n\in\mathbb{N}} \Sigma^\infty MGL_n[1/\beta] \to PMGL$ is an equivalence.*

Proof. Write

$$M := \bigvee_{n\in\mathbb{N}} \Sigma^\infty MGL_n[1/\beta] \to PMGL,$$

and consider the natural transformation of set-valued functors

$$\mathrm{Rings}(PMGL, -) \longrightarrow \mathrm{Rings}(M, -).$$

Given a ring spectrum R, we have seen that the set $\mathrm{Rings}(M, R)$ is naturally isomorphic to the set of collections $\{\vartheta_n\}_{n\in\mathbb{N}}$ with $\vartheta_n \in R^0(MGL_n)$ such that ϑ_{p+q} restricts to $\vartheta_p\vartheta_p$, ϑ_1 restricts to a unit in R^{-1}, and $\vartheta_0 = 1 \in R^0(S^0)$. Similarly, the set $\mathrm{Rings}(PMGL, R)$ is naturally isomorphic to the product of the set of units in R^{-1} and the set of collections $\{\theta_n\}_{n\in\mathbb{N}}$ with $R^n(MGL_n)$ such that θ_{p+q} restricts to $\theta_p\theta_q$, θ_1 restricts to the image of $1 \in R^0(S^0)$ in $R^1(S^1)$, and $\theta_0 = 1 \in R^0(S^0)$.

The map $\mathrm{Rings}(PMGL, R) \to \mathrm{Rings}(M, R)$ sends $\mu \in R^{-1}$ and $\theta_n \in R^n(MGL_n)$ to $\vartheta_n = \mu^n\theta_n$. We get a natural map back which sends $\vartheta_n \in R^0(MGL_n)$ to $\theta_n = \mu^{-n}\vartheta_n$, where $\mu \in R^{-1}$ in the unit corresponding to $\beta^*\vartheta_1 \in R^0(S^1)$. Clearly the composites are the respective identities, and we conclude that $M \to PMGL$ is an equivalence. □

3.4. $\Sigma^\infty_+ BGL[1/\beta]$ **is orientable.** Recall from [15] that, just as in the usual stable homotopy category, an orientation on a ring spectrum R is equivalent to a class in $R^1(MGL_1)$ which restricts, under the inclusion $i : S^1 \to MGL_1$ of the bottom cell, to the class in $R^1(S^1)$ corresponding to the unit $1 \in R^0(S^0)$ under the suspension isomorphism $R^0(S^0) \to R^1(S^1)$. Note also that in the case R is periodic with Bott element $\beta \in R^0(S^1)$, corresponding under the suspension isomorphism to the unit $\mu \in R^{-1}(S^0)$ with inverse $\mu^{-1} \in R^1(S^0)$, then the suspension isomorphism $R^0(S^0) \to R^1(S^1)$ sends 1 to $\mu^{-1}\beta$.

Now there's a canonical class $\theta_1 \in \Sigma^\infty_+ BGL[1/\beta]^1(MGL_1)$ such that

$$\mu^{-1}\beta = i^*\theta_1 \in \Sigma^\infty_+ BGL[1/\beta]^1(S^1).$$

Namely, set $\theta_1 := \mu^{-1}\vartheta_1$, where $\vartheta_1 \in \Sigma^\infty_+ BGL[1/\beta]^0(MGL_1)$ is the class of the composite

$$\Sigma^\infty MGL_1 \simeq \Sigma^\infty BGL_1 \longrightarrow \Sigma^\infty_+ BGL \longrightarrow \Sigma^\infty_+ BGL[1/\beta].$$

Then $\beta = i^*\mu\theta$, so $\mu^{-1}\beta = i^*\theta$.

Proposition 3.5. *There is a canonical ring map $\theta : PMGL \to \Sigma^\infty_+ BGL[1/\beta]$.*

Proof. The Thom class $\theta_1 \in \Sigma^\infty_+ BGL[1/\beta]^0(MGL_1)$ extends, as in [1] or [15], to a ring map $MGL \to \Sigma^\infty_+ BGL[1/\beta]$, and we have a canonical unit $\mu \in R^{-1}(S^0)$, the image of $\beta \in R^0(S^1)$ under the suspension isomorphism $R^0(S^1) \cong R^{-1}(S^0)$. □

Corollary 3.6. *There is a canonical ring map $\vartheta : \bigvee_n \Sigma^\infty MGL_n[1/\beta] \to \Sigma^\infty_+ BGL[1/\beta]$.*

Proof. We precompose the map from the previous Proposition 3.5 with the equivalence $\bigvee_n \Sigma^\infty MGL_n[1/\beta] \to PMGL$. □

3.5. ϑ **is an equivalence.** We analyze the effect of $\vartheta : \bigvee_n \Sigma^\infty MGL_n[1/\beta] \to \Sigma^\infty_+ BGL[1/\beta]$ on cohomology. To this end, fix an oriented periodic motivic ring spectrum R; we aim to show that the induced map

$$R^0(\Sigma^\infty_+ BGL[1/\beta]) \longrightarrow R^0(\bigvee_n \Sigma^\infty MGL_n[1/\beta])$$

is an isomorphism.

We begin with some algebra.

Lemma 3.7. *Let R be a commutative ring and $A = \operatorname{colim}_n A_n$ be a filtered commutative R-algebra such that $A_0 = R$ and each of the exact sequences*

$$0 \longrightarrow A_{n-1} \longrightarrow A_n \longrightarrow A_n/A_{n-1} \longrightarrow 0$$

is canonically split. Then there is a canonical R-module isomorphism

$$A = \operatorname{colim}_n A_n \cong \bigoplus_n A_n/A_{n-1}.$$

Moreover,

$$A \otimes_R A = \operatorname{colim}_{p,q} A_p \otimes_R A_q \cong \bigoplus_{p,q} A_p/A_{p-1} \otimes_R A_q/A_{q-1},$$

and the multiplication

$$A_p/A_{p-1} \otimes_R A_q/A_{q-1} \longrightarrow A_{p+q}/A_{p+q-1}$$

makes $A \cong \bigoplus_n A_n/A_{n-1}$ into an isomorphism of commutative R-algebras. □

Proposition 3.8. *There is a commuting square of R^0-module maps*

$$\begin{tikzcd} R^0(BGL) \arrow[r] \arrow[d] & \prod_n R^0(MGL_n) \arrow[d] \\ R^0(BGL \times BGL) \arrow[r] & \prod_{p,q} R^0(MGL_p \wedge MGL_q) \end{tikzcd},$$

in which the vertical maps are induced by the multiplication on BGL and $\bigvee_n MGL_n$, respectively, and the horizontal maps are isomorphisms.

Proof. Set $A := \operatorname{colim}_n \operatorname{Sym}^n_{R_0} R_0(\mathbb{P}^\infty)$, where the map $\operatorname{Sym}^{n-1}_{R_0} R_0(\mathbb{P}^\infty) \to \operatorname{Sym}^n_{R_0} R_0(\mathbb{P}^\infty)$ is induced by the the inclusion $R_0 \cong R_0(\mathbb{P}^0) \to R_0(\mathbb{P}^\infty)$. By the lemma, we have a commutative square

$$\begin{tikzcd} \bigoplus\limits_{p,q} \operatorname{Sym}^p R_0(\mathbb{P}^\infty)/\operatorname{Sym}^{p-1} R_0(\mathbb{P}^\infty) \underset{R_0}{\otimes} \operatorname{Sym}^q R_0(\mathbb{P}^\infty)/\operatorname{Sym}^{q-1} R_0(\mathbb{P}^\infty) \arrow[r] \arrow[d] & A \underset{R_0}{\otimes} A \arrow[d] \\ \bigoplus_n \operatorname{Sym}^n R_0(\mathbb{P}^\infty)/\operatorname{Sym}^{n-1} R_0(\mathbb{P}^\infty) \arrow[r] & A \end{tikzcd}$$

in which the vertical maps are multiplication and the horizontal maps are R_0-algebra isomorphisms. The desired commutative square is obtained by taking R_0-module duals. $\square$

Theorem 3.9. *The map of oriented periodic motivic ring spectra*

$$\vartheta : \bigvee_n \Sigma^\infty MGL_n[1/\beta] \to \Sigma^\infty_+ BGL[1/\beta]$$

is an equivalence.

Proof. We show that the induced natural transformation

$$\vartheta^* : \operatorname{Rings}(\Sigma^\infty_+ BGL[1/\beta], -) \longrightarrow \operatorname{Rings}(\bigvee_n \Sigma^\infty MGL_n[1/\beta], -)$$

is in fact a natural isomorphism. The result then follows immediately from Yoneda's Lemma.

Fix a ring spectrum R, and observe that, for another ring spectrum A, $\operatorname{Rings}(A, R)$ is the equalizer of the pair of maps from $R^0(A)$ to $R^0(A \wedge A) \times R^0(\mathbb{S})$ which assert the commutativity of the diagrams

$$\begin{tikzcd} A \wedge A \arrow[r] \arrow[d] & A \arrow[d] \\ R \wedge R \arrow[r] & R \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbb{S} \arrow[r] \arrow[dr] & A \arrow[d] \\ & R \end{tikzcd}.$$

Given a map $\beta : \mathbb{S}^1 \to A$, the set $\operatorname{Rings}(A[1/\beta], R)$ is the equalizer of the pair of maps from $\operatorname{Rings}(A, R) \times R^0(\mathbb{S}^{-1})$ to $R^0(\mathbb{S})$ which assert that the ring map $A \to R$ is such that there's a spectrum map $\mathbb{S}^{-1} \to R$ for which the product

$$\mathbb{S}^1 \wedge \mathbb{S}^{-1} \longrightarrow A \wedge R \longrightarrow R \wedge R \longrightarrow R$$

is equivalent to the unit $\mathbb{S} \to R$. Putting these together, we may express $\operatorname{Rings}(A[1/\beta], R)$ as the equalizer of natural pair of maps from $R^0(A) \times R^0(\mathbb{S}^{-1})$ to $R^0(A \wedge A) \times R^0(\mathbb{S}) \wedge R^0(\mathbb{S})$.

We therefore get a map of equalizer diagrams

$$\begin{tikzcd} R^0(BGL)\times R^0(\mathbb{S}^{-1}) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & R^0(BGL\times BGL)\times R^0(\mathbb{S})\times R^0(\mathbb{S}) \arrow[d] \\ \prod_n R^0(MGL_n)\times R^0(\mathbb{S}^{-1}) \arrow[r, shift left] \arrow[r, shift right] & \prod_{p,q} R^0(MGL_p\wedge MGL_q)\times R^0(\mathbb{S})\times R^0(\mathbb{S}) \end{tikzcd},$$

the equalizer of which ϑ^*. Now if R does *not* admit the structure of a $PMGL$-algebra, then clearly there cannot be any ring maps from either of the $PMGL$-algebras $\bigvee_n \Sigma^\infty MGL_n[1/\beta]$ of $\Sigma^\infty_+ BGL[1/\beta]$. Hence we may assume that R is also an oriented periodic ring spectrum, in which case Proposition 3.8 implies that the vertical maps are isomorphisms. □

Corollary 3.10. *The map of periodic oriented motivic ring spectra*

$$\theta : PMGL \to \Sigma^\infty_+ BGL[1/\beta]$$

is an equivalence.

Proof. $\bigvee_n \Sigma^\infty MGL_n[1/\beta] \to PMGL$ is an equivalence. □

4. Algebraic K-Theory

4.1. **The representing spectrum.** Let $BGL_{\mathbb{Z}} \simeq \mathbb{Z}\times BGL$ denote the group completion of the motivic E_∞ monoid

$$BGL_{\mathbb{N}} := \coprod_{n\in\mathbb{N}} BGL_n.$$

Given a motivic space X, write $K^0(X) := \pi_0 \operatorname{map}_S(X, BGL_{\mathbb{Z}})$. If $S = \operatorname{spec}\mathbb{Z}$ and X is a scheme, this agrees with the homotopy algebraic K-theory of X as defined by Weibel [27]; if in addition X is regular, this also agrees with Thomason-Trobaugh algebraic K-theory of X [22]. As the name suggests, homotopy algebraic K-theory is a homotopy invariant version of the Thomason-Trobaugh algebraic K-theory, and homotopy invariance is of course a prerequisite for any motivic cohomology theory.

Proposition 4.1 (Quillen)**.** *Let $p : V \to X$ be a vector bundle of rank r over a quasicompact scheme X and let $q : \mathbb{P}(V) \to X$ be the associated projective bundle. Then the map $K^{*,0}(X)^r \to K^{*,0}(\mathbb{P}(V))$ which sends $(a_0,\dots,a_{r-1}) \in K^{*,0}(X)^r$ to $q^*(a_0)+q^*(a_1)z+\cdots+q^*(a_n)z^n \in K^{*,0}(\mathbb{P}(V))$, where $z \in K^{0,0}(\mathbb{P}(V))$ is the class of the tautological line bundle over $\mathbb{P}(V)$, is an isomorphism.* □

The projective bundle theorem implies the following fundamental facts.

Proposition 4.2. *Let X be a motivic space. Then tensor product of vector bundles induces an isomorphism*

$$K^0(\mathbb{P}^1) \underset{K^0}{\otimes} K^0(X) \to K^0(\mathbb{P}^1\times X)$$

of abelian groups. □

Proposition 4.3. *Let $L \in K^0(\mathbb{P}^1)$ denote the class of the canonical line bundle, and let $K^0[\lambda] \to K^0(\mathbb{P}^1)$ be the ring map which sends λ to $L-1$. Then λ^2 goes to zero in $K^0(\mathbb{P}^1)$, and the resulting map $K^0[\lambda]/(\lambda^2) \to K^0(\mathbb{P}^1)$ is an isomorphism.* □

Proposition 4.4. *There is an isomorphism of split short exact sequences of $K^0(X)$-modules*

$$\begin{tikzcd}
\lambda K^0(X)[\lambda]/(\lambda^2) \arrow[r] \arrow[d] & K^0(X)[\lambda]/(\lambda^2) \arrow[r] \arrow[d] & K^0(X) \arrow[d] \\
K^0((\mathbb{P}^1,\mathbb{P}^0)\wedge(X,0)) \arrow[r] & K^0(\mathbb{P}^1\times X) \arrow[r] & K^0(\mathbb{P}^0\times X)
\end{tikzcd} ;$$

in particular, $K^0((\mathbb{P}^1,\mathbb{P}^0)\wedge(X,0))$ is isomorphic to $K^0(X)$. □

We deduce the Bott periodicity theorem as a corollary.

Corollary 4.5 (Motivic Bott Periodicity)**.** *The adjoint*

$$(BGL_{\mathbb{Z}}, BGL_0) \longrightarrow \Omega(BGL_{\mathbb{Z}}, BGL_0),$$

of the map $\Sigma(BGL_{\mathbb{Z}}, BGL_0) \to (BGL_{\mathbb{Z}}, BGL_0)$, classifying the tensor product $(L-1)\otimes V$, where $V \to BGL_{\mathbb{Z}}$ is the universal virtual vector bundle, is an equivalence. □

The motivic Bott periodicity theorem allows us to define a motivic spectrum representing algebraic K-theory. Define a sequence of pointed spaces $K(n)$ by

$$K(n) := (BGL_{\mathbb{Z}}, BGL_0)$$

for all $n \in \mathbb{N}$. By Theorem 4.5, each $K(n)$ comes equipped with an equivalence

$$K(n) = (BGL_{\mathbb{Z}}, BGL_0) \to \Omega(BGL_{\mathbb{Z}}, BGL_0) = \Omega K(n+1),$$

making $K := (K(0), K(1), \ldots)$ into a motivic spectrum.

4.2. **A map $\Sigma^\infty_+\mathbb{P}^\infty[1/\beta] \to K$.** Let $\beta : \mathbb{P}^1 \to \mathbb{P}^\infty$ be the map classifying the tautological line bundle on $\mathbb{P}^1$. We construct a ring map $\Sigma^\infty_+\mathbb{P}^\infty \to K$ which sends β to a unit in K, thus yielding a ring map $\Sigma^\infty_+\mathbb{P}^\infty[1/\beta] \to K$.

Ring maps $\Sigma^\infty_+\mathbb{P}^\infty \to K$ are adjoint to monoidal maps $B\mathbb{G} \to GL_1K$, the multiplicative monoid of units (up to homotopy) in the ring space $\Omega^\infty K \simeq BGL_{\mathbb{Z}}$. Since $\pi_0 BGL_{\mathbb{Z}}$ contains a copy of $\mathbb{Z}$, the multiplicative units contain the subgroup $\{\pm 1\} \to \mathbb{Z}$, giving a map $\{\pm 1\} \times BGL \to GL_1K$. But the inclusion $BGL_1 \to BGL$ is monoidal with respect to the multiplicative structure on BGL, so we get a monoidal map

$$\mathbb{P}^\infty \simeq BGL_1 \longrightarrow \{+1\} \times BGL \longrightarrow GL_1K$$

and therefore a ring map $\Sigma^\infty_+\mathbb{P}^\infty \to K$.

Proposition 4.6. *The class of the composite*

$$\Sigma^\infty S^1 \longrightarrow \Sigma^\infty_+\mathbb{P}^\infty \longrightarrow K$$

is equal to that of the K-theory Bott element β_K, i.e. the class of the reduced tautological line bundle $L-1$ on $\mathbb{P}^1$.

Proof. The map $\Sigma^\infty_+\mathbb{P}^\infty \to K$ classifies the tautological line bundle on $\mathbb{P}^\infty$, so the pointed version $\Sigma^\infty\mathbb{P}^\infty \to K$ corresponds to the reduced tautological line bundle on $\mathbb{P}^\infty$. This restricts to the reduced tautological line bundle on $\mathbb{P}^1$. □

Corollary 4.7. *There's a canonical ring map $\psi : \Sigma^\infty_+\mathbb{P}^\infty[1/\beta] \to K$.*

4.3. **The addition on $BGL_{\mathbb{Z}}$.**

Proposition 4.8. *The identity map $BGL_{\mathbb{Z}} \to \Omega^\infty K$ is additive.*

Proof. In other words, the addition on $K^0(-)$ is induced by the addition of vector bundles. □

Corollary 4.9. *Let R be an oriented periodic ring spectrum. Given $f : BGL_{\mathbb{Z}} \to \Omega^\infty R$, the resulting map $\Omega_R f : BGL_{\mathbb{Z}} \to \Omega BGL_{\mathbb{Z}} \to \Omega^{\infty+1} R \to \Omega^\infty R$ is additive.*

Proof. Ωf is additive with respect to the addition induced by the infinite loop space structure on $BGL_{\mathbb{Z}}$. But by Proposition 4.8, this is the same as the usual addition on $BGL_{\mathbb{Z}}$. □

4.4. **A useful splitting.** Given a motivic spectrum R and a pointed motivic space X, we write $X \wedge R$ for the spectrum $\Sigma^\infty X \wedge R$ and R^X for the spectrum of maps from $\Sigma^\infty X$ to R.

Proposition 4.10. *Let R be an oriented periodic ring spectrum and let Z be an arbitrary spectrum. Then the natural map $R^0(\mathbb{P}^\infty_+) \otimes_{R^0} R^0(Z) \to R^0(\mathbb{P}^\infty_+ \wedge Z)$ is an isomorphism.*

Proof. Inductively, we get a morphism of exact sequences

$$\begin{array}{ccccc}
R^0(Z) & \longrightarrow & R^0(Z)^{n+1} & \longrightarrow & R^0(Z)^n \\
\downarrow & & \downarrow & & \downarrow \\
R^0((\mathbb{P}^n, \mathbb{P}^{n-1}) \wedge Z) & \longrightarrow & R^0(\mathbb{P}^n_+ \wedge Z) & \longrightarrow & R^0(\mathbb{P}^{n-1}_+ \wedge Z)
\end{array}$$

in which the left and right vertical maps are isomorphisms. It follows that the middle map is also an isomorphism. We are therefore reduced to a short exact sequence

$$0 \longrightarrow \lim_n{}^1 R^0(S^{1,0} \otimes Z)^{n+1} \longrightarrow R^0(\mathbb{P}^n_+ \otimes Z) \longrightarrow \lim_n R^0(Z)^{n+1} \longrightarrow 0.$$

Since the maps $R^0(S^{1,0} \otimes Z)^{n+1} \to R^0(S^{1,0} \otimes Z)^n$ are surjective, the $\lim^1$ term vanishes and we are left with an isomorphism $R^0(\mathbb{P}^\infty_+) \otimes_{R^0} R^0(Z) \cong R^0(\mathbb{P}^\infty_+ \wedge Z)$. □

Corollary 4.11. *Let R be an oriented periodic motivic ring spectrum. Then the additive maps from $BGL_{\mathbb{Z}}$ to $\Omega^\infty R$ define a canonical section $s : R^{\mathbb{P}^\infty_+} \to R^{BGL_{\mathbb{Z}}}$ of the restriction $r : R^{BGL_{\mathbb{Z}}} \to R^{\mathbb{P}^\infty_+}$ induced by the inclusion $\mathbb{P}^\infty_+ \simeq BGL_0 + BGL_1 \to BGL_{\mathbb{Z}}$.*

Proof. Set $Y = R^{BGL_{\mathbb{Z}}}$ and $Z = R^{\mathbb{P}^\infty_+}$. By Proposition 2.26, the additive maps from $BGL_{\mathbb{Z}}$ to $\Omega^\infty R$ define a canonical section $\pi_0 Z \to \pi_0 Y$ of the surjection $\pi_0 Y \to \pi_0 Z$. We must lift this to a map of spectra $s : Z \to Y$.

By Proposition 4.10, we have isomorphisms $R^0(\mathbb{P}^\infty_+ \otimes Z) \cong R^0(Z) \otimes_{\pi_0 R} \pi_0 Z \cong Z^0(Z)$. Combined with the section $\pi_0 Y \to \pi_0 Z$, this induces a map

$$Z^0(Z) \to R^0(Z) \otimes_{\pi_0 R} \pi_0 Z \to R^0(Z) \otimes_{\pi_0 R} \pi_0 Y \to Y^0(Z).$$

Take $s \in Y^0(Z)$ to be the image of $1 \in Z^0(Z)$ under this map. □

4.5. **Comparing $R^0(K)$ and $R^0(L)$.** .

Proposition 4.12. *Let R be an oriented motivic ring spectrum, let $\alpha \in \pi_1 R$ be a unit, and let $\alpha_* : R \to \Sigma^{-1} R$ denote the automorphism induced by multiplication by α. Then*

$$R^K \simeq \mathrm{holim}_n \{ \cdots \xrightarrow{g} R^{\mathbb{P}^\infty_+} \xrightarrow{g} R^{\mathbb{P}^\infty_+} \},$$

where the map $g : R^{\mathbb{P}^\infty_+} \to R^{\mathbb{P}^\infty_+}$ *sends* $\lambda : \Sigma^\infty_+\mathbb{P}^\infty \to R$ *to* $\alpha_*^{-1} \circ \Sigma^{-1}\lambda \circ \beta_*$*, the composite*

$$\Sigma^\infty_+\mathbb{P}^\infty \to \Sigma^{-1}\Sigma^\infty_+\mathbb{P}^\infty \to \Sigma^{-1}R \to R.$$

Proof. Once again, set $Y = R^{BGL_{\mathbb{Z}}}$ and $Z = R^{\mathbb{P}^\infty_+}$. By definition, $\Omega^\infty\Sigma^n K = BGL_{\mathbb{Z}}$ for all n, so

$$R^K \simeq \operatorname{holim}_n \Sigma^n Y \simeq \operatorname{holim}_n Y,$$

where the map on the right is induced by $\Sigma^n\alpha_*^n : \Sigma^n R \to R$. By Proposition 4.11, the map $r : Y \to Z$ is a split surjection with section $s : Z \to Y$.

Define $f : Y \to Y$ by sending $\sigma : \Sigma^\infty BGL_{\mathbb{Z}} \to R$ to $\alpha_*^{-1} \circ \Sigma^{-1}\sigma \circ \beta_* : \Sigma^\infty BGL_{\mathbb{Z}} \to R$, and note that the square

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "s"'] & Z \arrow[d, "s"] \\ Y \arrow[r, "f"] & Y \end{tikzcd},$$

commutes. We claim that $f = s \circ g \circ r : Y \to Y$; indeed, by Corollary 4.9, f factors through the additive maps from $BGL_{\mathbb{Z}}$ to R, which is to say it factors through s. Hence

$$f = s \circ r \circ f = s \circ g \circ r,$$

and the result follows from Lemma 2.4. □

For the remainder of this section, let L denote the localized motivic ring spectrum

$$L := \Sigma^\infty_+\mathbb{P}^\infty[1/\beta].$$

Corollary 4.13. *Let R be an oriented motivic ring spectrum, let $\alpha \in \pi_1 R$ be a unit, and let $\alpha_* : R \to \Sigma^{-1}R$ denote the automorphism induced by multiplication by α. Then $\psi : L \to K$ induces an isomorphism $\psi^* : R^0(K) \to R^0(L)$.*

Proof. This is immediate from Proposition 4.12 above, since the spectrum of spectrum maps from L to R admits precisely the same description as that of the spectrum of spectrum maps from K to R. Indeed,

$$L \simeq \operatorname{colim}_n\{\Sigma^\infty_+\mathbb{P}^\infty \xrightarrow{\beta_*} \Sigma^{-1}\Sigma^\infty_+\mathbb{P}^\infty \xrightarrow{\beta_*} \cdots\},$$

and we see that

$$R^L \simeq \operatorname{holim}_n\{\cdots \xrightarrow{g} R^{\mathbb{P}^\infty_+} \xrightarrow{g} R^{\mathbb{P}^\infty_+}\},$$

where the map $g : R^{\mathbb{P}^\infty_+} \to R^{\mathbb{P}^\infty_+}$ sends $\lambda : \Sigma^\infty_+\mathbb{P}^\infty \to R$ to $\alpha_*^{-1} \circ \Sigma^{-1}\lambda \circ \beta_*$, the composite

$$\Sigma^\infty_+\mathbb{P}^\infty \to \Sigma^{-1}\Sigma^\infty_+\mathbb{P}^\infty \to \Sigma^{-1}R \to R,$$

just as above. □

By the homotopy category of oriented periodic spectra, we mean the full subcategory of the homotopy category of spectra on the oriented periodic objects. Note that, according to this definition, maps need not preserve the orientations or even the ring structure.

Proposition 4.14. *ψ induces an isomorphism $\psi^* : [K, -] \to [L, -]$ of functors from the homotopy category of orientable periodic spectra to abelian groups.*

Proof. Let R be an orientable periodic spectrum. Then

$$\psi^* : [K, R] = R^0(K) \to R^0(L) = [L, R]$$

is an isomorphism by Corollary 4.13, and this isomorphism is natural in spectrum maps $R \to R'$, provided of course that R' is also orientable and periodic. □

Theorem 4.15. *The ring map $\psi : L \to K$ is an equivalence.*

Proof. Let $\varphi^* : [L, -] \to [K, -]$ be the isomorphism inverse to the isomorphism $\psi^* : [K, -] \to [L, -]$ of Proposition 4.14, and let $\varphi : K \to L$ be the map obtained by applying φ^* to the identity $1 \in [L, L]$. It follows from the Yondeda lemma that φ^* is precomposition with φ. The equations $\varphi^* \circ \psi^* = 1_K^*$ and $\psi^* \circ \varphi^* = 1_L^*$ imply that $\psi \circ \varphi = 1_K$ and $\varphi \circ \psi = 1_L$ in the homotopy category of orientable periodic spectra, and therefore that $\psi \circ \varphi = 1_K$ and $\varphi \circ \psi = 1_L$ is the homotopy category of spectra. Hence $\psi : L \to K$ is an equivalence with inverse $\varphi : K \to L$. □

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